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DEGENERATE PRINCIPAL SERIES REPRESENTATIONS FOR $U(n,n)$

BY

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ABSTRACT

For a quadratic extension E/F of a nonarchimedean local field of characteristic other than 2, let $G = U(n,n)$ be the quasisplit unitary group of rank n, and let P be the maximal parabolic subgroup of G which stabilizes a maximal isotropic subspace. Then P has a Levi decomposition $P = MN$ with $M \simeq \mathrm{GL}(n,E)$. In this paper, the points of reducibility and composition series of the degenerate principal series $I_n(s,\chi)$ defined by characters of M are determined completely. The constituents arising as theta lifts of characters of $U(m)$'s are identified and their behavior under the intertwining operator $M(s,\chi)$: $I_n(s,\chi) \to I_n(-s,\check{\chi})$ is described. The case $E = F \oplus F$ and $G \simeq \mathrm{GL}(2n,F)$ is included.

1. Introduction

The degenerate principal series representations of split classical groups over local fields which are induced from a character of the maximal parabolic subgroup with abelian unipotent radical turn out to play a central role in the local and

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global theta correspondence, and, ultimately, in a number of interesting problems in arithmetic. In the local theory, these representations are a key ingredient in understanding the basic structure of the local theta correspondence. Globally, these degenerate principal series play an essential part in the proof of an extended Siegel-Weil formula [20], [16]. For these and other applications, it is essential to know (i) all points of reducibility, (ii) the complete composition series at each such point, (iii) the identification of the constituents as arising from the local theta correspondence, (iv) the behavior of constituents under normalized intertwining operators, and certain other information about generalized Whittaker models and non-singularity. In the case of $U(n,n)$ or $\mathrm{GL}(2n)$ over a nonarchimedean local field, all of this information is obtained in the present paper. In [7], our results are used to investigate dichotomy phenomena in the local theta correspondence for unitary groups, to calculate local L-factors, and to interpret local ϵ-factors for supercuspidal representations of such groups. Our results will also allow the proof of the extended Siegel-Weil formula of [20] to be carried over to the unitary case. The expected formula has already found several applications [6], [3].

At the moment, our knowledge of the degenerate principal series in the general case is incomplete, although there are a number of partial results available. For example, fairly good information in the case of $\mathrm{Sp}(n)$ over a local field was obtained in [19]. The nonarchimedean symplectic case was also considered in [5], [11], [32]. Some results in the case of orthogonal groups were obtained in [12]. Finally, information in the archimedean case may be found in [18], for $\mathrm{Sp}(n,\mathbb{R})$ (items (i), (iii), and (iv), above) and [21], for $U(n,n)$ (items (i) and (ii)). The combination of work of Lee [21] and Zhu [37] should finish (iii) and (iv) for $U(n,n)$, and should complete (iii) in the case of $\mathrm{Sp}(n,\mathbb{R})$.

The methods of this paper are substantially those of [19], so that we will sometimes omit details. On the other hand, [19] relies on a number of results which exist in the literature only for the symplectic group and often without published proof. Thus, for example, we have given a complete discussion of the poles and normalization of the intertwining operator in the unitary case, based on an analogue of the method of [31]. Some details of this computation play an essential role in [7], in particular in the definition of the ϵ-factor by the doubling method, and in the interpretation of the resulting root number in terms of the local theta correspondence.

We now give a more precise description of our results.

Let F be a nonarchimedean local field not of characteristic 2, and let E/F be a quadratic extension. We write $x \mapsto \bar{x}$ for the action of the non-trivial Galois automorphism of E/F. Write $\mathcal{O}_E$ for the ring of integers of E, $\mathcal{P}_E$ for the maximal ideal of $\mathcal{O}_E$, and q_E for the order of the finite field $\mathcal{O}_E/\mathcal{P}_E$. We choose, once and for all, a generator π_E for $\mathcal{P}_E$, and normalize $|\,|_E$ via $|\pi_E|_E = q_E^{-1}$. Similar notations will be used for F, save that we will not fix a particular choice of π_F.

Let G be the isometry group of the skew Hermitian form defined on the space of row vectors $W = E^{2n}$ via

$$\langle u, v\rangle = uJ\,{}^t\bar{v}, \qquad \text{with } J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}. \tag{1.1}$$

Thus $g \in \mathrm{GL}(2n, E)$ belongs to G if and only if $gJ\,{}^t\bar{g} = J$. G has a maximal parabolic subgroup $P = MN$ given by

$$\begin{aligned} M &= \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix} \mid a \in \mathrm{GL}(n,E) \right\}, \\ N &= \left\{ n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} \mid b = {}^t\bar{b} \in M(n,E) \right\}. \end{aligned} \tag{1.2}$$

For a given unitary character χ of $E^\times$ and for $s \in \mathbb{C}$, we consider the induced representation of G realized by the space of functions

(1.3)

$$\begin{aligned} I(s,\chi) = \{\Phi\colon G \to \mathbb{C} \text{ smooth } \mid \Phi(n(b)m(a)g) = \chi(\det(a))|\det(a)|_E^{s+\frac{n}{2}}\Phi(g) \\ \text{for all } g \in G,\ m(a) \in M, \text{ and } n(b) \in N\}, \end{aligned}$$

where G acts by right translation. This is normalized induction, so that $\mathrm{Re}(s) = 0$ is the unitary axis.

Since we wish to describe the quasi-character $\chi|\,|_E^s$ by the pair (s,χ), it will often be convenient to normalize s and χ to make this correspondence unique. Rather than requiring that $\chi(\pi_E) = 1$ for our fixed choice of π_E, as in [33] or [19], it will be more convenient to choose s and χ so that the pair satisfies one or both of the properties below:

$$\chi(\pi_E \cdot \bar{\pi}_E) = 1, \tag{P1}$$

$$\mathrm{Im}(s) \in \left[0, \frac{\pi}{\log(q_E)}\right). \tag{P2}$$

The pair (s, χ) is easily made to satisfy (P1) by a shift in s, and we can also require that $\mathrm{Im}(s) \in [0, \frac{2\pi}{\log(q_E)})$ by merely shifting s by an element of $\frac{2\pi i}{\log(q_E)} \cdot \mathbb{Z}$, which does not change $|\ |_E^s$. This still leaves some ambiguity, however: if χ_1 and χ_2 satisfy (P1), then $\chi_1 |\ |_E^{s_1} = \chi_2 |\ |_E^{s_2}$ implies only that $s_1 - s_2 \in \frac{\pi i}{\log(q_E)} \cdot \mathbb{Z}$. So for our given $\chi |\ |_E^s$, we can replace χ by $\chi |\ |_E^{\frac{\pi i}{\log(q_E)}}$ and s by $s - \frac{\pi i}{\log(q_E)}$, if necessary, so that s satisfies (P2) and χ still satisfies (P1). This then makes the pair (s, χ) unique. If χ satisfies (P1), we will say that χ is normalized, while if both (P1) and (P2) hold, then the pair (s, χ) is normalized.

We will frequently write $\check{\chi}$ for the character $\check{\chi}(x) = \chi(\bar{x})^{-1}$. Notice that $\chi = \check{\chi}$ if and only if either $\chi|_{F^\times} = 1$ or $\chi|_{F^\times} = \epsilon_{E/F}$, this last denoting the unique non-trivial quadratic character of $F^\times$ with kernel equal to $N_F^E(E^\times)$.

The points of reducibility of $I(s, \chi)$ are given by the following theorem:

THEOREM 1.1: *Let (s, χ) be normalized as above.*

(1) *If $\chi \neq \check{\chi}$, then $I(s, \chi)$ is irreducible for all s.*

(2) *If $\chi|_{F^\times} = 1$, then $I(s, \chi)$ is irreducible except when*

$$s \in \left\{ -\frac{n}{2}, 1 - \frac{n}{2}, 2 - \frac{n}{2}, \ldots, \frac{n}{2} \right\}.$$

(3) *If $\chi|_{F^\times} = \epsilon_{E/F}$, then $I(s, \chi)$ is irreducible except when*

$$s \in \left\{ -\frac{n-1}{2}, 1 - \frac{n-1}{2}, 2 - \frac{n-1}{2}, \ldots, \frac{n-1}{2} \right\}.$$

Note that, in each case, the points of reducibility consist either of integral or half-integral points, and that $I(0, \chi)$ is reducible if and only if $\chi|_{F^\times} = 1$ and n is even, or $\chi|_{F^\times} = \epsilon_{E/F}$ and n is odd.

The statement of the theorem above may be slightly misleading, in that, if one fixes a character χ satisfying (P1) and allows s to vary over all of $\mathbb{C}$, then the pattern of points of reducibility is a bit more complicated than even the usual $\frac{2\pi i}{\log(q_E)} \cdot \mathbb{Z}$ discrepancy would allow. For example, when E/F is unramified, $\chi |\ |_E^{\frac{\pi i}{\log(q_E)}}$ restricted to F equals $(\chi|_{F^\times}) \cdot \epsilon_{E/F}$, and in normalizing s, cases (2) and (3) of the theorem may be permuted. On the other hand, when E/F is ramified, $|\ |_E^{\frac{\pi i}{\log(q_E)}} = 1$ on F, and all translates by $\frac{\pi i}{\log(q_E)} \cdot \mathbb{Z}$ of the values of s given above are points of reducibility.

At each of the points of reducibility described above, the constituents of $I(s, \chi)$ arise from the images of certain representations of G associated to Hermitian forms via the Weil or oscillator representation. If V is a non-degenerate Hermitian vector space with $\dim_E(V) = m \geq 1$, then we can view $U(V) \times G$ as a dual reductive pair in the larger symplectic group $\mathrm{Sp}(4mn, F)$. If $\chi|_{F^\times} = \epsilon_{E/F}^m$, then χ can be used to construct a splitting of the inverse image of G sitting in the metaplectic cover of $\mathrm{Sp}(4mn, F)$ ([15]). The Weil representation then leads to a representation ω of G in the Schwartz-Bruhat space $\mathcal{S}(V^n)$, and in fact, for $\varphi \in \mathcal{S}(V^n)$, the function $\omega(g)\varphi(0)$ lies in the space $I(s_0, \chi)$ for $s_0 = \frac{m-n}{2}$. We denote the image of $\mathcal{S}(V^n)$ in $I(s_0,\chi)$ by $R(V,\chi)$. If $\chi|_{F^\times} = 1$, then χ (which acts as a character of P) extends uniquely to a character χ_G of G. In this case, we let $R(0,\chi) \subset I(-\frac{n}{2},\chi)$ be the $\mathbb{C}$-linear span of χ_G. These spaces form the constituents of $I(s_0,\chi)$ as follows.

THEOREM 1.2: *Suppose that $\chi|_{F^\times} = \epsilon_{E/F}^m$ and $s_0 = \frac{m-n}{2}$ for some $m \geq 0$. This accounts for all points of reducibility of $I(s_0,\chi)$. Whenever $m \geq 1$, let V_1 and V_2 be the two inequivalent non-degenerate Hermitian vector spaces over E of dimension m. These are distinguished by $\det(Q_i) \in F^\times/N_F^E(E^\times)$, where Q_i is some choice of Hermitian matrix realizing the form on V_i.*

(1) *If $m = 0$, then $R(0,\chi)$ is the unique irreducible G-submodule contained in $I(-\frac{n}{2},\chi)$, and $I(-\frac{n}{2},\chi)/R(0,\chi)$ is also irreducible.*

(2) *If $1 \leq m \leq n$, so that $-\frac{n}{2} < s_0 \leq 0$, then $R(V_1,\chi)$ and $R(V_2,\chi)$ are irreducible and inequivalent, $R(V_1,\chi) \oplus R(V_2,\chi)$ is a submodule of $I(s_0,\chi)$, and*

$$I(s_0,\chi)/(R(V_1,\chi) \oplus R(V_2,\chi))$$

is irreducible. In particular, if $m = n$, then $R(V_1,\chi) \oplus R(V_2,\chi) = I(0,\chi)$.

(3) *If $n < m < 2n$, so that $0 < s_0 < \frac{n}{2}$, then $R(V_1,\chi)$ and $R(V_2,\chi)$ are distinct maximal submodules of $I(s_0,\chi)$, so that $I(s_0,\chi) = R(V_1,\chi) + R(V_2,\chi)$. Also, $R(V_1,\chi) \cap R(V_2,\chi)$ is the unique irreducible submodule of $I(s_0,\chi)$.*

(4) *If $m = 2n$, then let V_1 be the split Hermitian space of dimension m, and V_2 the other space of the same dimension. In this case, $R(V_2,\chi)$ is the unique maximal submodule of $I(\frac{n}{2},\chi)$, and $R(V_2,\chi)$ is irreducible with codimension 1. In addition, $R(V_1,\chi) = I(\frac{n}{2},\chi)$.*

(5) *Finally, if $m > 2n$, so that $I(s_0,\chi)$ is irreducible by Theorem 1.1, then the submodules $R(V_i,\chi)$ are both equal to all of $I(s_0,\chi)$.*

As with the analogous representations for the symplectic group ([19]), the intertwining operator

$$M(s,\chi)\colon I(s,\chi) \longrightarrow I(-s,\check{\chi}) \tag{1.4}$$

plays a central role in our calculations, and it is essential to normalize this operator. This problem is considered in section 3. The key point is to consider the functional equation

$$W_\beta(-s) \circ M(s,\chi) = \gamma(s,\chi,\beta) \cdot W_\beta(s), \tag{1.5}$$

((3.5) in section 3) where $W_\beta(s)$ is the generalized Whittaker functional. Using 'Rallis's Lemma', the computation of the factor $\gamma(s,\chi,\beta)$ is reduced (Proposition 3.1) to that of the factor in the intermediate functional equation (3.8) of the family of zeta integrals attached to the prehomogeneous vector space of Hermitian forms $\mathrm{Herm}_n(E)$. This intermediate functional equation is then determined, (3.9), by the method of [31]. It should be noted that the precise information about $\gamma(s,\chi,\beta)$ obtained in section 3 plays an essential role in the investigation of theta dichotomy and epsilon factors in [7]. Also note that, in the present paper, we normalize the intertwining operator $M(s,\chi)$ so that $M^*(s,\chi)$ has no poles and is never identically zero, rather than by a condition on the local functional equation. Once $M^*(s,\chi)$ is defined ((3.27), (3.28)), it relates the quotients of $I(s,\chi)$ on one side of the unitary axis $\mathrm{Re}(s)=0$ to the submodules on the other. The kernel and image of $M^*(s,\chi)$ are determined for all s and χ (Propositions 5.8, 5.10, 6.4, and 6.6).

In section 2, we review the rough information about points of reducibility and constituents of our degenerate principal series representations which can be obtained by a direct application of the Jacquet functor. Section 3 contains the normalization of the intertwining operator, by the method just described. In section 4, we use the Weil representation to construct submodules at various points of possible reducibility. In section 5, we compute the Jacquet functor with respect to a maximal parabolic subgroup which stabilizes a line. This computation reveals an inductive structure and allows us to obtain a lot of information about the composition series and behavior of constituents under the normalized intertwining operator. Finally, in section 6, we compute the exponents of the various constituents coming from the Weil representation. These allow us to complete our picture, and, in particular, to show the irreducibility of the 'third piece'.

In global applications, it is also necessary to consider the case $E = F \oplus F$ with 'Galois' automorphism $x = (x_1, x_2) \mapsto \bar{x} = (x_2, x_1)$. The definitions given above can be carried over to this case, and, after a suitable idempotent projection, we arrive at $G = \mathrm{GL}(2n, F)$ and maximal parabolic $P = MN$ given by

$$(1.6) \qquad M = \{m(a,d) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathrm{GL}(n,F)\},$$
$$N = \{n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} \mid b \in M(n,F)\}.$$

Given two unitary characters σ_1, σ_2 of $F^\times$, we let $\sigma = (\sigma_1, \sigma_2)$, and write (s, σ) for the quasi-character of M defined by

$$(1.7) \qquad (s,\sigma)(m(a,d)) = \sigma_1(a)|a|_F^s \, \sigma_2(d)|d|_F^{-s},$$

determinant being understood, as usual. The local representation

$$(1.8) \qquad I(s,\sigma) = \mathrm{Ind}_P^G(\sigma_1|\,|_F^s \otimes \sigma_2|\,|_F^{-s})$$

is of a type studied by Bernstein and Zelevinsky in [1], [2], and [36]. Suppose now that we fix a generator π_F of the maximal ideal $\mathcal{P}_F$ of $\mathcal{O}_F$. By adjusting s by the square root of $\sigma_1(\pi_F)/\sigma_2(\pi_F)$, we may assume that (s,σ) is normalized to satisfy $\sigma_1(\pi_F) = \sigma_2(\pi_F)$ and $\mathrm{Im}(s) \in [0, \frac{2\pi}{\log(q_F)})$. We have the following result.

THEOREM 1.3: *Suppose that (s,σ) is normalized as above.*

(1) *If $\sigma_1 = \sigma_2$ and $s \in \{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots, \pm\frac{n}{2}\}$, then $I(s,\sigma)$ is reducible as a $\mathrm{GL}(2n,F)$ module.*
(2) *For all other pairs (s,σ), $I(s,\sigma)$ is irreducible.*
(3) *Suppose that $\sigma = \sigma_1 = \sigma_2$ (in the obvious abuse of notation) and let $s_0 = \frac{m-n}{2}$ for $0 \le m \le 2n$, $m \ne n$, so that (s_0,σ) is a point of reducibility for $I(s,\sigma)$. Then $I(s_0,\sigma)$ has a unique irreducible submodule A, and the quotient $I(s,\sigma)/A$ is also irreducible, and is not isomorphic to A. For $1 \le k \le n$, let $P_{n+k,n-k}$ be the standard maximal parabolic of G with Levi factor isomorphic to $\mathrm{GL}(n+k,F) \times \mathrm{GL}(n-k,F)$.*
   (a) *If $n < m \le 2n$, so that $0 < s_0 \le \frac{n}{2}$, then the irreducible quotient of $I(s_0,\sigma)$ is isomorphic to $\mathrm{Ind}_{P_{m,2n-m}}^G(\sigma)$.*
   (b) *On the other side of the unitary axis, if $0 \le m < n$ (so that $-\frac{n}{2} \le s_0 < 0$), then the irreducible subrepresentation of $I(s_0,\sigma)$ is isomorphic to $\mathrm{Ind}_{P_{2n-m,m}}^G(\sigma)$.*

Notice that the irreducible quotient at $s_0 > 0$ *is isomorphic to the irreducible submodule at* $-s_0$ *(see also (5) below), and that both are unitarizable.*

(4) *These constituents are related to the Weil representation as follows. For* $\sigma = \sigma_1 = \sigma_2$ *and* $s_0 = \frac{m-n}{2}$ *(with* $1 \leq m$*), there is a construction of a dual pair* $H = \mathrm{GL}(m, F)$ *and* $G = \mathrm{GL}(2n, F)$ *in* $\mathrm{Sp}(4mn, F)$*. The appropriate Weil representation restricts to a representation of* $H \times G$*, and there is an injection of the* H*-coinvariants into a subspace of* $\mathrm{Ind}_P^G(|\,|_F^{s_0} \otimes |\,|_F^{-s_0})$*. We twist this subspace by* σ*, and call the resulting submodule* $R_n(m, \sigma) \subset I(s_0, \sigma)$*.*

(a) *If* $m \geq n$*, then* $R_n(m, \sigma) = I(s_0, \sigma)$*.*

(b) *If* $1 \leq m < n$*, then* $R_n(m, \sigma)$ *is the unique irreducible (unitarizable) submodule of* $I(s_0, \sigma)$*.*

(c) *If* $m = 0$*, then we may set* $R_n(0, \sigma) = \mathbb{C} \cdot \sigma \subset I(-\frac{n}{2}, \sigma)$*. This is the unique irreducible submodule in this case.*

(5) *There is a standard* G*-intertwining operator* $M(s, \sigma)$*:* $I(s, \sigma) \to I(-s, \check{\sigma})$*, where* $\sigma = (\sigma_1, \sigma_2)$ *is again arbitrary, and* $\check{\sigma} = (\sigma_2, \sigma_1)$*, defined by*

$$M(s, \sigma)\Phi(g) = \int_{M(n,F)} \Phi(w_n n(b) g)\, db \tag{1.9}$$

(see equation (2.2) for w_n*). This converges for* $\mathrm{Re}(s) > \frac{n}{2}$*. We normalize this operator by setting* $M^*(s, \sigma) = a(s, \chi_\sigma)^{-1} M(s, \sigma)$*, where* $\chi_\sigma = \sigma_1/\sigma_2$ *is a character of* $F^\times$*, and*

$$a(s, \chi_\sigma) = \zeta_F(2s, \chi_\sigma)\zeta_F(2s-1, \chi_\sigma) \cdots \zeta_F(2s-(n-1), \chi_\sigma)$$

is a product of Tate zeta functions (notation as in section 3 below). Then for any σ*,* $M^*(s, \sigma)$ *has an analytic continuation to the* s*-plane, and it is never identically the zero operator. When* (s, σ) *is not a point of reducibility,* $M^*(s, \sigma)$ *is an isomorphism. When* (s, σ) *is a point of reducibility,* $M^*(s, \sigma)$ *has the unique irreducible submodule of* $I(s, \sigma)$ *as kernel, and maps the unique irreducible quotient of* $I(s, \sigma)$ *to the irreducible submodule of* $I(-s, \sigma)$*.*

The proof of this result is sketched in section 7.

In a future paper, we plan to exploit the results just described to establish the extended Siegel-Weil formula of [20] in the case of unitary groups. For a quadratic extension E/F of number fields, and for a non-degenerate Hermitian space V with $\dim_E V = m < n$, the Siegel-Weil formula will identify the restricted tensor product of the local representations $R_n(V,\chi)$ (at nonarchimedean places of F which are not split in E) and $R_n(m,\sigma)$ (at nonarchimedean places of F which are split in E) as a space of automorphic forms generated by the residues of the Siegel Eisenstein series on $U(n,n)$. We must, of course, include components, arising in the work of Lee [21], at the archimedean places. Moreover, this same automorphic representation will be generated by (regularized) theta integrals, and the coincidence of the two definitions amounts to an identification of theta integrals and (residues of) Eisenstein series. This application is one of the main motivations for the present paper.

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2. Jacquet modules

First, we define a bit more notation. If $H \supset K$ are l-groups ([1]) and V is an H-module, then let

$$V(K) = \mathrm{span}_{\mathbb{C}}\{k \cdot v - v \mid k \in K \text{ and } v \in V\}, \tag{2.1}$$

so that $V_K = V/V(K)$ is the Jacquet module of V with respect to K, a $\mathrm{Norm}_H(K)$-module. In particular, for any G-module V, we can regard V_N as an M-module (or as a P-module with N acting trivially). We will frequently regard characters of $E^\times$ as defining characters of $\mathrm{GL}(n,E)$ via $\chi(a) = \chi(\det(a))$ for $a \in \mathrm{GL}(n,E)$. The relative norm and trace of the extension E/F will be denoted by N_F^E and T_F^E, respectively, while the trace of a matrix $x \in M(n,E)$ will be written $\mathrm{tr}(x)$.

LEMMA 2.1: *The Jacquet module $I(s,\chi)_N$ has an M-stable filtration*

$$I(s,\chi)_N = I^0 \supset I^1 \supset \cdots \supset I^n \supset I^{n+1} = 0$$

with successive quotients

$$Z^r(s,\chi) = I^r/I^{r+1} \simeq \mathrm{Ind}_{Q_r}^{\mathrm{GL}(n)}(\xi_r),$$

where $Q_r \subset \mathrm{GL}(n,E)$ is the maximal parabolic subgroup of the form

$$Q_r = \left\{ \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \middle| a \in \mathrm{GL}(n-r), b \in \mathrm{GL}(r) \right\}$$

and ξ_r is the character of Q_r given by

$$\xi_r\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} = \chi(a)\check{\chi}(b)|a|_E^{s+\frac{n-r}{2}}|b|_E^{-s+\frac{r}{2}}.$$

The induction above is normalized by

$$\delta_{Q_r}^{\frac{1}{2}}\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} = \left(\frac{\Delta(\mathrm{GL}(n))}{\Delta(Q_r)}\right)^{\frac{1}{2}}\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} = |a|_E^{\frac{r}{2}}|b|_E^{\frac{r-n}{2}},$$

so that $f \in \mathrm{Ind}_{Q_r}^{\mathrm{GL}(n)}(\xi_r)$ satisfies $f(pg) = \xi_r(p)\delta_{Q_r}^{\frac{1}{2}}(p)f(g)$ for all $p \in Q_r$, $g \in$ $\mathrm{GL}(n,E)$.

Sketch of proof: First of all, we choose double coset representatives w_r for $P\backslash G/P$: for $0 \le r \le n$, let

$$w_r = \begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_r \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & -1_r & 0 & 0 \end{pmatrix}. \tag{2.2}$$

Then the relative Bruhat decomposition holds: $G = \coprod_{j=0}^{n} Pw_jP$. Fixing s and χ, let $J^0 = I(s,\chi)$, and for $1 \le r \le n+1$, $J^r = \{f \in J^0 \mid f = 0 \text{ on } Pw_{r-1}P\}$. Alternately, define $J^{n+1} = 0$, and $J^r = \{f \in I(s,\chi) \mid \mathrm{supp}(f) \subset \coprod_{j=r}^{n} Pw_jP\}$ for $0 \le r \le n$. Also define subgroups of N for $0 \le r \le n$ by

$$N_r = \left\{ n\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a = {}^t\bar{a} \in M(r,E)\right\}.$$

We then have a P-intertwining map

$$\begin{aligned} J^r &\longrightarrow \mathrm{Ind}_{Q_r}^{\mathrm{GL}(n)}(\xi_r), \\ \Phi &\longmapsto \left\{ m(a) \mapsto \int_{N_r} \Phi(w_r n m(a))dn \right\} \end{aligned} \tag{2.3}$$

(N acting trivially on the right-hand space). Modulo checking convergence, it is easily seen that this is well-defined, and that the map factors through J^{r+1}. Gustafson checks in a similar situation [5] that the integral converges, that the map is surjective, and that the kernel of the map $J^r/J^{r+1} \to \mathrm{Ind}_{Q_r}^{\mathrm{GL}(n)}(\xi_r)$ equals the space $(J^r/J^{r+1})(N)$. By the exactness of the N-Jacquet functor (see [1]), we then have an M-module isomorphism of J_N^r/J_N^{r+1} with the space $\mathrm{Ind}_{Q_r}^{\mathrm{GL}(n)}(\xi_r)$. Setting $I^r = J_N^r$ for each r finishes the proof. ∎

The proofs of the following two results are the same as those of Propositions 2.2 and 2.3 in [19], and hence are omitted.

PROPOSITION 2.2: *Suppose that (s,χ) is normalized. Then*

(1)

$$\dim \mathrm{Hom}_G(I(s,\chi), I(-s,\check{\chi})) \leq \begin{cases} 2 & \text{if } \chi = \check{\chi} \text{ and } s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{n-1}{2}\}, \\ 1 & \text{otherwise.} \end{cases}$$

(2)

$$\dim \operatorname*{Hom}_G(I(s,\chi), I(s,\chi)) \leq \begin{cases} 2 & \text{if } \chi = \check{\chi} \text{ and } s = 0, \\ 1 & \text{otherwise.} \end{cases}$$

PROPOSITION 2.3: *Let χ be normalized, and suppose that π is a G-submodule of $I(s,\chi)$. If $\chi = \check{\chi}$ and*

$$s \in -\frac{n-r}{2} + \frac{\pi i}{r \cdot \log(q_E)} \cdot \mathbb{Z} \quad \text{for some } r \text{ with } 1 \leq r \leq n$$

then

$$\dim \operatorname*{Hom}_G(\pi, I(s,\chi)) \leq 2,$$

and hence $I(s,\chi)$ has at most two irreducible submodules. Otherwise,

$$\dim \operatorname*{Hom}_G(\pi, I(s,\chi)) = 1,$$

and $I(s,\chi)$ has at most one irreducible submodule.

3. The intertwining operator and points of reducibility

In order to limit the possible points of reducibility of $I(s,\chi)$, we need to do some background work on the intertwining operator

$$M(s,\chi)\colon I(s,\chi) \longrightarrow I(-s,\check{\chi}). \tag{3.1}$$

As in the symplectic case, this is defined, for $\mathrm{Re}(s) > \frac{n}{2}$, by the integral

$$(M(s,\chi)\Phi)(g) = \int_N \Phi(w_n n g, s)\, dn, \tag{3.2}$$

and by meromorphic continuation otherwise. We must specify the normalization of the Haar measure on N. Let $X = \{x \in M(n,E) \mid x = {}^t\bar{x}\}$, and note that $N \simeq X$. Choose a non-trivial character ψ of F^+, and let dx be the self-dual measure on X with respect to the Fourier transform defined by

$$\hat{\phi}(y) = \int_X \phi(x)\psi(\mathrm{tr}(xy))\, dx, \tag{3.3}$$

for $\phi \in \mathcal{S}(X)$, the space of locally constant, compactly supported functions on X. We then have $\hat{\hat{\phi}}(x) = \phi(-x)$. The measure used in the definition of $M(s,\chi)$ is $dn(x) = dx$. Note that this measure, and hence the operator $M(s,\chi)$, depends on the choice of ψ.

Our immediate goal is to normalize $M(s,\chi)$ so that it is entire and non-vanishing (as an operator), and to determine when the normalized operator is injective. To do this we consider the generalized Whittaker functional $W_\beta(s)$ on $I(s,\chi)$ as in Karel's paper [13]. This functional is defined, for $\beta \in X$ and for $\mathrm{Re}(s) > \frac{n}{2}$, by the integral

$$(W_\beta(s)\Phi)(g) = \int_X \Phi(w_n n(b)g, s)\psi(\mathrm{tr}(b\beta))\,db. \tag{3.4}$$

If $\det(\beta) \neq 0$, $W_\beta(s)$ has an entire analytic continuation, and by uniqueness, satisfies a functional equation

$$W_\beta(-s) \circ M(s,\chi) = \gamma(s,\chi,\beta) \cdot W_\beta(s) \tag{3.5}$$

for some meromorphic function $\gamma(s,\chi,\beta)$. Our normalization of $M(s,\chi)$ will depend on an explicit computation of $\gamma(s,\chi,\beta)$, which is the analogue, in our situation, of the local factor of Shahidi [28], [4].

As we will see in a moment, the local factor $\gamma(s,\chi,\beta)$ is closely related to the family of zeta integrals attached to a certain prehomogeneous vector space. Let $Y = \{x \in X \mid \det(x) \neq 0\}$, so that the triple $(\mathrm{GL}(n,E), X, Y)$ forms a prehomogeneous vector space, as in Igusa [8]. An element $g \in \mathrm{GL}(n,E)$ acts on $x \in X$ via $x \mapsto gx\,{}^t\bar{g}$, and this action divides Y into two orbits: $Y = Y_1 \coprod Y_2$, where we may take $Y_1 = \{x \in X \mid \det(x) \in N_F^E(E^\times)\}$, and $Y_2 = Y \sim Y_1$ [10]. There is a $\mathrm{GL}(n,E)$-invariant measure on Y given by

$$d^\times x = \frac{dx}{|\det(x)|_F^{n}}. \tag{3.6}$$

For a unitary character τ of $F^\times$ and for $\phi \in \mathcal{S}(X)$, we can then define zeta integrals via

$$Z_j(s,\tau,\phi) = \int_{Y_j} \tau(x)|x|_F^s\phi(x)\,d^\times x, \tag{3.7}$$

for $j = 1, 2$. Similarly, we define $Z(s,\tau,\phi)$ by integrating over Y rather than Y_j. These integrals converge for $\mathrm{Re}(s) \gg 0$ and have meromorphic analytic continuations. There is an intermediate functional equation of the form

$$Z(s,\tau,\phi)=\sum_{j=1}^{2}e_j(s,\tau)Z_j(n-s,\tau^{-1},\hat{\phi}), \tag{3.8}$$

where the $e_j(s,\tau)$ are meromorphic in s. (The complete functional equation relates $Z_i(s,\tau,\phi)$ to $Z_j(n-s,\tau^{-1},\hat{\phi})$ for $j=1$, 2.) For any $\beta\in Y_j$, it is convenient to write $e_\beta(s,\tau)$ in place of $e_j(s,\tau)$, although the factor depends only on the orbit of β. In the appendix to the current section, these factors have been computed to be

$$e_\beta(s,\tau)=\gamma_E(\bar{\psi}\circ N_F^E)^{\frac{(n-1)n}{2}}\epsilon_{E/F}(\beta)^{n-1}\prod_{r=0}^{n-1}\rho_F(s+r-(n-1),\tau\cdot\epsilon_{E/F}^{r}). \tag{3.9}$$

Here γ_E is the Weil index (an eighth root of unity) of the indicated character of second degree of E^+, and $\rho_F(s,\tau)$ is the γ-factor from the local functional equation for $\mathrm{GL}(1,F)$ in Tate's thesis. In brief, if we define $\zeta_F(s,\tau)$ to be the zeta function of F given by

$$\zeta_F(s,\tau)=\begin{cases}\frac{1}{1-\tau(\pi_F)q_F^{-s}} & \text{if } \tau \text{ is unramified,}\\ 1 & \text{if } \tau \text{ is ramified,}\end{cases} \tag{3.10}$$

then

$$\rho_F(s,\tau)=\frac{\zeta_F(s,\tau)}{\zeta_F(1-s,\tau^{-1})}\cdot(\text{exponential factors}). \tag{3.11}$$

For more detail, see the appendix to this section. Now we solve for $\gamma(s,\chi,\beta)$.

Proposition 3.1: *For any $\beta\in X$ with $\det(\beta)\neq 0$, the factors $e_\beta(s,\chi)$ and $\gamma(s,\chi,\beta)$ from the respective functional equations of $Z(s,\chi)$ and $W_\beta(s)$ are related by*

$$e_\beta(2s,\chi)=\chi(\beta)|\beta|_E^{s}\,\gamma(s,\chi,\beta).$$

Note that (s,χ) need not be normalized here, and also that both sides depend only on $\chi|_{F^\times}$.

Proof: Given $\beta\in X$ with $\det(\beta)\neq 0$, choose $\varphi\in\mathcal{S}(X)$ satisfying $\hat{\varphi}(\beta)\neq 0$, and define a section $\Phi_\varphi(s)\in I(s,\chi)$ by requiring that $\mathrm{supp}(\Phi_\varphi)\subset Pw_nN$ and

$$\Phi_\varphi(w_n n(b),s)=\varphi(b) \qquad \text{for } b\in X. \tag{3.12}$$

Notice that

$$(3.13)\qquad (W_\beta(s)\Phi_\varphi)(e) = \int_X \Phi_\varphi(w_n n(b), s)\psi(\mathrm{tr}(b\beta))\,db = \int_X \varphi(b)\psi(\mathrm{tr}(b\beta))\,db = \hat{\varphi}(\beta).$$

Let $\Psi(-s) = M(s,\chi)\Phi_\varphi \in I(-s,\check{\chi})$. Then by (3.5) and (3.13),

$$(3.14)\qquad (W_\beta(-s)\Psi)(e) = \gamma(s,\chi,\beta)\hat{\varphi}(\beta).$$

By the definition of $W_\beta(-s)$, for $\mathrm{Re}(s) \ll 0$ we also have

$$(3.15)\qquad (W_\beta(-s)\Psi)(e) = \int_X \Psi(w_n n(b), -s)\psi(\mathrm{tr}(b\beta))\,db,$$

while for $\mathrm{Re}(s) \gg 0$,

$$(3.16)\qquad \Psi(w_n n(b), -s) = \int_X \Phi_\varphi(w_n n(x) w_n n(b), s)\,dx.$$

But for $x \in Y$,

$$(3.17)\qquad w_n n(x) w_n = \begin{pmatrix} -1 & 0 \\ x & -1 \end{pmatrix} = \begin{pmatrix} -x^{-1} & 1 \\ 0 & -x \end{pmatrix} w_n n(-x^{-1})$$
$$\Longrightarrow w_n n(x) w_n n(b) = \begin{pmatrix} -x^{-1} & 1 \\ 0 & -x \end{pmatrix} w_n n(b - x^{-1}).$$

We use the notation $\phi_b(y) = \phi(b+y)$ for any $\phi \in \mathcal{S}(X)$ and $b \in X$. Then for $\mathrm{Re}(s) \gg 0$,

$$(3.18)\qquad \begin{aligned} \Psi(w_n n(b), -s) &= \int_X \chi(-x^{-1})|-x^{-1}|_E^{s+\frac{n}{2}} \varphi(b - x^{-1})\,dx \\ &= \int_X \chi(-x^{-1})|-x^{-1}|_E^{s} \varphi_b(-x^{-1})\,d^\times x \\ &= \int_X \chi(x)|x|_E^{s} \varphi_b(x)\,d^\times x \\ &= Z(2s,\chi,\varphi_b) \end{aligned}$$

(note that $|\,|_E = |\,|_F^2$), this final equation holding now for almost all s by the continuation of $Z(2s)$. So for $\mathrm{Re}(s) \ll 0$ we have

$$(3.19)\qquad \begin{aligned} (W_\beta(-s)\Psi)(e) &= \int_X Z(2s,\chi,\varphi_b)\psi(\mathrm{tr}(b\beta))\,db \\ &= \sum_{j=1}^{2} e_j(2s,\chi)\mathcal{I}_j, \end{aligned}$$

where $\mathcal{I}_j$ is the integral

$$\mathcal{I}_j = \int_X Z_j(n-2s, \chi^{-1}, \widehat{(\varphi_b)})\psi(\mathrm{tr}(b\beta))\, db. \tag{3.20}$$

Now let $B = \{b \in X \mid b_{ii} \in \mathcal{O}_F \text{ and } b_{ij} \in \mathcal{O}_E \text{ for all } i < j\}$. For $r \in \mathbb{Z}$ and some choice of generator π of $\mathcal{P}_F$, set $B_r = \pi^{-r} \cdot B$, so that $X = \bigcup_{r=0}^{\infty} B_r$. Then for $\mathrm{Re}(s) \ll 0$,

(3.21)

$$\begin{aligned}\mathcal{I}_j &= \lim_{r\to\infty} \int_{B_r} \int_{Y_j} \chi(x)^{-1}|x|_F^{n-2s}\widehat{(\varphi_b)}(x)\, d^\times x \quad \psi(\mathrm{tr}(b\beta))\, db \\ &= \lim_{r\to\infty} \int_{Y_j} \chi(x)^{-1}|x|_F^{-2s}\hat{\varphi}(x)\left[\int_{B_r} \psi(\mathrm{tr}(b(\beta - x)))db\right] dx\end{aligned}$$

since $\widehat{(\varphi_b)}(x) = \psi(-\mathrm{tr}(bx))\hat{\varphi}(x)$ and both integrals are over compact sets. But

$$\int_{B_r} \psi(\mathrm{tr}(b(\beta - x)))\, db = \hat{\phi}_r(x-\beta), \tag{3.22}$$

where $\phi_r(x)$ is the characteristic function of B_r. There is some constant $c \in \mathbb{Z}$ such that $\mathrm{supp}(\hat{\phi}_r) \subset B_{c-r}$, and we note that $\{B_{c-r}\}_{r=0}^{\infty}$ gives a system of compact open neighborhoods of 0 in X. Since χ, $|\,|_F$, and $\hat{\varphi}$ are locally constant, there are two cases:

(1) If $\beta \in Y_j$ (an open subset of X), then for r large enough,

(3.23)

$$\begin{aligned}\int_{Y_j} \chi(x)^{-1}|x|_F^{-2s}\hat{\varphi}(x)\hat{\phi}_r(x-\beta)\, dx &= \int_{\beta+B_{c-r}} \chi(x)^{-1}|x|_F^{-2s}\hat{\varphi}(x)\hat{\phi}_r(x-\beta)\, dx \\ &= \chi(\beta)^{-1}|\beta|_F^{-2s}\hat{\varphi}(\beta) \int_{\beta+B_{c-r}} \hat{\phi}_r(x-\beta)\, dx \\ &= \chi(\beta)^{-1}|\beta|_E^{-s}\hat{\varphi}(\beta) \int_X \hat{\phi}_r(x)\, dx \\ &= \chi(\beta)^{-1}|\beta|_E^{-s}\hat{\varphi}(\beta)\hat{\hat{\phi}}_r(0) \\ &= \chi(\beta)^{-1}|\beta|_E^{-s}\hat{\varphi}(\beta).\end{aligned}$$

(2) If, on the other hand, $\beta \notin Y_j$, then the integral is clearly 0.

So $\mathcal{I}_j$ equals either 0 or $\chi(\beta)^{-1}|\beta|_E^{-s}\hat{\varphi}(\beta)$, and we have

$$(W_\beta(-s)\Psi)(e) = e_\beta(2s,\chi)\chi(\beta)^{-1}|\beta|_E^{-s}\hat{\varphi}(\beta), \tag{3.24}$$

which, combined with equation (3.14), yields

$$\gamma(s,\chi,\beta) = \chi(\beta)^{-1}|\beta|_E^{-s} e_\beta(2s,\chi). \qquad \blacksquare \tag{3.25}$$

Notice that $\gamma(s,\chi,\beta)$ depends on β in a fairly trivial way. In order to separate out the behavior of the zeros and poles of $\gamma(s,\chi,\beta)$, we will write

$$\gamma(s,\chi) = \gamma(s,\chi,I_n). \tag{3.26}$$

Now we can perform a fairly detailed analysis of the intertwining operator $M(s,\chi)$. Let

$$a(s,\chi) = \prod_{j=0}^{n-1} \zeta_F(2s+j-(n-1),\chi\cdot\epsilon_{E/F}^j), \tag{3.27}$$

which is just the numerator of $\gamma(s,\chi)$ when it is written as a quotient of products of zeta functions (see equation (3.9) and Proposition 3.1 above). We normalize $M(s,\chi)$ by setting

$$M^*(s,\chi) = \frac{1}{a(s,\chi)}\cdot M(s,\chi). \tag{3.28}$$

PROPOSITION 3.2: *The normalized intertwining operator $M^*(s,\chi)$ is entire, and for any given value of $s\in\mathbb{C}$, there exists a holomorphic section $\Phi(s)\in I(s,\chi)$ such that $M^*(s,\chi)\Phi(s)$ is non-zero. Furthermore, if (s,χ) is normalized, and $(s,\chi)\notin R$, where*

$$R = \{(s,\chi)\in\mathbb{C}\times\widehat{E^\times} \mid \gamma(s,\chi)\gamma(-s,\check{\chi}) = 0, \text{ or } \chi=\check{\chi} \text{ and } s=0\},$$

then $M^*(s,\chi)$ is an injective operator, as is $M^*(-s,\check{\chi})$, by symmetry.

Proof: The first step in the proof that $M^*(s,\chi)$ is entire is a standard check (as in [24]) that the analytic properties (i.e., degrees of poles and zeros) of the family of *sections*

$$M(s,\chi)\Phi(s) \tag{3.29}$$

(as $\Phi(s)$ ranges over all holomorphic sections of $I(s,\chi)$) coincide with those of the family of *functions*

$$M(s,\chi)\Phi_\varphi(w_n,s) \tag{3.30}$$

(as φ ranges over $\mathcal{S}(X)$). This fact is sometimes referred to as Rallis's Lemma [29]. See the proof of Proposition 3.1 for the definition of Φ_φ. Next, setting $b = 0$ in equation (3.18) of that proof yields

$$M(s,\chi)\Phi_\varphi(w_n,s) = Z(2s,\chi,\varphi). \tag{3.31}$$

Note that, in fact, the right-hand term above depends only on χ restricted to $F^\times$ as does $a(s,\chi)$. So the first two assertions of Proposition 3.2 reduce to analogous assertions about the functions

$$\frac{1}{a(s,\chi)} \cdot Z(2s,\chi,\varphi). \tag{3.32}$$

Since $a(s,\chi)$ is also the common numerator of the factors $e_j(2s,\chi)$ occurring in the functional equation of $Z(2s,\chi,\varphi)$, the proof from this point on follows that of the theorem on p. 106 of [24]. This concludes the proofs of the first two assertions.

Now, given that $M^*(s,\chi)$ is entire, the operator

$$M^*(-s,\check{\chi}) \circ M^*(s,\chi)\colon I(s,\chi) \longrightarrow I(s,\chi) \tag{3.33}$$

is well-defined, and, since we are assuming that $(s,\chi) \notin R$, the operator must in fact be a scalar, by Proposition 2.2. By using the functional equation (3.5) of $W_\beta(s)$ twice, we have

$$W_\beta(s) \circ M^*(-s,\check{\chi}) \circ M^*(s,\chi) = \frac{\gamma(s,\chi,\beta)}{a(s,\chi)} \cdot \frac{\gamma(-s,\check{\chi},\beta)}{a(-s,\check{\chi})} \cdot W_\beta(s). \tag{3.34}$$

If we apply this to a section of the form $\Phi_\varphi(s)$, noting that $(W_\beta(s)\Phi_\varphi)(e) = \hat{\varphi}(\beta)$ can be chosen to be non-zero, we can conclude that

$$M^*(-s,\check{\chi}) \circ M^*(s,\chi) = \frac{\gamma(s,\chi,\beta)}{a(s,\chi)} \cdot \frac{\gamma(-s,\check{\chi},\beta)}{a(-s,\check{\chi})} \neq 0. \tag{3.35}$$

This proves the injectivity of $M^*(s,\chi)$. ∎

We can now state the theorem giving the possible points of reducibility of $I(s,\chi)$.

THEOREM 3.3: *Suppose that (s,χ) is normalized, and let*

$$R = \{(s,\chi) \in \mathbb{C} \times \widehat{E^\times} \mid \gamma(s,\chi)\gamma(-s,\check{\chi}) = 0, \text{ or } \chi = \check{\chi} \text{ and } s = 0\}.$$

If $(s,\chi) \notin R$, then $I(s,\chi)$ is irreducible.

Proof: Suppose that $(s,\chi) \notin R$, and that $A \subset I(s,\chi)$ is a proper irreducible submodule. Then we have an exact sequence

$$0 \longrightarrow A \xrightarrow{\imath} I(s,\chi) \longrightarrow B \longrightarrow 0 \tag{3.36}$$

with A and B non-zero. Taking contragredients, and noting that $\widetilde{I(s,\chi)} \simeq I(-s,\chi^{-1})$, we obtain

$$0 \longrightarrow \widetilde{B} \longrightarrow I(-s,\chi^{-1}) \xrightarrow{\tilde{\imath}} \widetilde{A} \longrightarrow 0. \tag{3.37}$$

Now let $\delta \in \mathrm{GL}_F(E^{2n})$ be given by

$$(x,y)\delta = (\bar{x}, -\bar{y}) \qquad \text{for } (x,y) \in E^n \oplus E^n = E^{2n}, \tag{3.38}$$

and note that conjugation by δ preserves G. If the action of G on A is given by π, let π^δ be defined by $\pi^\delta(g) = \pi(\delta^{-1}g\delta)$, and denote this new representation by A^δ. By Theorem II.1, p. 91 of [23], since A is irreducible and admissible, $\widetilde{A} \simeq A^\delta$. Also notice that $I(s,\chi)^\delta \simeq I(s,\check{\chi}^{-1})$ via $\Phi \mapsto \Phi^\delta$, where $\Phi^\delta(g) = \Phi(\delta^{-1}g\delta)$. This follows from the fact that, if we represent $g \in G$ as $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ relative to the complete polarization $E^{2n} = E^n \oplus E^n$, then $\delta^{-1}g\delta = \begin{pmatrix} \bar{a} & -\bar{b} \\ -\bar{c} & \bar{d} \end{pmatrix}$. Hence we may define a mapping

$$T \in \operatorname*{Hom}_G(I(-s,\chi^{-1}), I(s,\check{\chi}^{-1})) \tag{3.39}$$

via

$$I(-s,\chi^{-1}) \xrightarrow{\tilde{\imath}} \widetilde{A} \simeq A^\delta \hookrightarrow I(s,\chi)^\delta \simeq I(s,\check{\chi}^{-1}). \tag{3.40}$$

Clearly $\ker(T) = \widetilde{B}$, and $0 \neq \widetilde{B} \subsetneq I(-s,\chi^{-1})$.

Now $(s,\chi) \notin R$ implies that $(-s,\chi^{-1}) \notin R$ (as R only depends on $\chi|_{F^\times}$), and so $M^*(-s,\chi^{-1})$ is injective by the preceding Proposition. Since T is non-zero and non-injective, the dimension of $\mathrm{Hom}_G(I(-s,\chi^{-1}), I(s,\check{\chi}^{-1}))$ must be at least 2, and so it follows that $\chi = \check{\chi}$ and $s \in \{0, -\frac{1}{2}, -1, \ldots, \frac{1-n}{2}\}$ by Proposition 2.2.

Next, we repeat the argument using a non-zero irreducible submodule $C \subset \ker T$. This yields an operator $T' \in \mathrm{Hom}_G(I(s,\chi), I(-s,\check{\chi}))$ which is again non-zero and non-injective. Hence T' is not a multiple of $M^*(s,\chi)$, so that the dimension of the given Hom space is at least 2, and again we conclude that $\chi = \check{\chi}$ and $s \in \{0, \frac{1}{2}, 1, \ldots, \frac{n-1}{2}\}$. But now we have shown that $(s,\chi) = (0,\check{\chi})$, which contradicts $(s,\chi) \notin R$. ∎

For convenience, we list the pairs $(s,\chi) \in R$ in a more elementary form. First, note the following.

LEMMA 3.4: *Suppose that χ is normalized. Then either*

(1) $\chi = \check{\chi}$, *in which case* $\chi|_{F^\times} = 1$ *or* $\chi|_{F^\times} = \epsilon_{E/F}$, *or*

(2) $\chi \neq \check{\chi}$, *in which case both* $\chi|_{F^\times}$ *and* $(\chi|_{F^\times}) \cdot \epsilon_{E/F}$ *are ramified characters of* $F^\times$.

Proof: The first assertion is trivial, so suppose that $\chi \neq \check{\chi}$. First assume that E/F is an unramified extension. Then 1 and $\epsilon_{E/F}$ are the only unramified quadratic characters of $F^\times$. It is easy to see that if $\chi|_{F^\times}$ were unramified, it would also be quadratic due to the normalization of χ, and so it would have to equal one of 1 or $\epsilon_{E/F}$. Since this is not the case, $\chi|_{F^\times}$ must be ramified. The same reasoning shows that $(\chi|_{F^\times}) \cdot \epsilon_{E/F}$ is also ramified. Suppose next that E/F is ramified. In this situation, if $\chi|_{F^\times}$ were unramified, it would be trivial, since $\pi_F = N_F^E(\pi_E)$ generates $\mathcal{P}_F$. Hence $\chi|_{F^\times}$ must be ramified, and similarly for $(\chi|_{F^\times}) \cdot \epsilon_{E/F}$. ∎

LEMMA 3.5: *Suppose that χ is normalized. If $\chi = \check{\chi}$, then let $\chi|_{F^\times} = \epsilon_{E/F}^m$, where $m = 0$ or 1.*

(1) *Suppose that E/F is an unramified extension, and that $\chi = \check{\chi}$. Then $\gamma(s,\chi) = 0$ when*

$$s \in \frac{n+2}{2} - k + \frac{\pi i}{\log(q_E)} \cdot (m + 2\mathbb{Z}) \qquad \text{for } 1 \leq k \leq \left[\frac{n+1}{2}\right],$$

or when

$$s \in \frac{n+1}{2} - k + \frac{\pi i}{\log(q_E)} \cdot (m + 1 + 2\mathbb{Z}) \qquad \text{for } 1 \leq k \leq \left[\frac{n}{2}\right].$$

(2) *Suppose that E/F is ramified, and $\chi = \check{\chi}$. If $m = 0$, then $\gamma(s,\chi) = 0$ when*

$$s \in \frac{n+2}{2} - k + \frac{\pi i}{\log(q_E)} \cdot \mathbb{Z} \qquad \text{for } 1 \leq k \leq \left[\frac{n+1}{2}\right].$$

If $m=1$, then $\gamma(s,\chi)=0$ when

$$s \in \frac{n+1}{2} - k + \frac{\pi i}{\log(q_E)} \cdot \mathbb{Z} \qquad \text{for } 1 \leq k \leq \left[\frac{n}{2}\right].$$

(3) *If $\chi \neq \check{\chi}$, then $\gamma(s,\chi)$ has no zeros.*

This is an easy computation using equation (3.9), Proposition 3.1, and Lemma 3.4.

The list of possible points of reducibility is much simplified by assuming that (s,χ) is normalized. As noted after Theorem 1.1, this has the slight disadvantage that one cannot fix χ and consider when reducibility occurs as $\chi|\,|_E^s$ varies over all quasi-characters in the equivalence class of χ.

THEOREM 3.6: *Let (s,χ) be normalized.*

(1) *If $\chi \neq \check{\chi}$, then $I(s,\chi)$ is irreducible for all s.*

(2) *If $\chi|_{F^\times}=1$, then $I(s,\chi)$ is irreducible except possibly when*

$$s=0 \qquad \text{or}$$
$$s=\pm\left(\frac{n+2}{2}-k\right) \quad \text{for some } k \text{ satisfying } 1 \leq k \leq \left[\frac{n+1}{2}\right].$$

(3) *If $\chi|_{F^\times}=\epsilon_{E/F}$, then $I(s,\chi)$ is irreducible except possibly when*

$$s=0 \qquad \text{or}$$
$$s=\pm\left(\frac{n+1}{2}-k\right) \quad \text{for some } k \text{ satisfying } 1 \leq k \leq \left[\frac{n}{2}\right].$$

Appendix to Section 3

Here we will solve for the factors $e_\beta(s,\tau)$ in the intermediate functional equation (3.8). The computation is along the same lines as that in [31]. Let all notation be as given earlier. For brevity, we write N and T for the relative norm and trace, respectively, of the extension E/F.

First, recall that for $\varphi \in \mathcal{S}(F)$ and for any character τ of $F^\times$, Tate's zeta function is given by

$$\zeta(\varphi,\tau|\,|_F^s) = \int_{F^\times} \varphi(x)\tau(x)|x|_F^s\, d^\times x. \tag{3a.1}$$

It satisfies a functional equation

$$\zeta(\varphi,\tau|\,|_F^s) = \rho_F(\tau|\,|_F^s)\cdot\zeta(\hat{\varphi},\tau^{-1}|\,|_F^{1-s}), \tag{3a.2}$$

where $\hat{\varphi}$ (depending on the choice of ψ) was defined earlier. As in [33], $\rho_F(\tau|\,|_F^s)$ is easily computed to be

$$\rho_F(\tau|\,|_F^s) =$$

$$\left[q_F^{-\frac{t}{2}} \sum_{\epsilon\in\mathcal{U}_F/(1+\mathcal{P}_F^t)} \tau(\epsilon\pi_F^{-D-f})\bar{\psi}(\epsilon\pi_F^{-D-f})\right] q_F^{(D+t)(s-\frac{1}{2})} \cdot \frac{\zeta_F(s,\tau)}{\zeta_F(1-s,\tau^{-1})}, \tag{3a.3}$$

where $\mathcal{P}_F^t$ is the conductor of τ, and D is maximal so that $\psi \equiv 1$ on $\mathcal{P}_F^{-D}$. Also, $\mathcal{P}_E^f$ is the conductor of the character $\tau\circ N$ of $E^\times$, so that $\tau\circ N = 1$ on $1+\mathcal{P}_E^f$. Note that ρ_F depends on the choice of ψ, and that the term in brackets [] above is a Gauss sum having absolute value 1.

The computation of $e_\beta(s,\tau)$ begins with the following:

LEMMA 3A.1: *For $\beta\in Y$ and $\mathrm{Re}(s)\gg 0$, we have*

$$e_\beta(s,\tau) = \tau(\beta)|\beta|_F^s \int_C \tau(x)|x|_F^s\bar{\psi}(\mathrm{tr}(x\beta))\,d^\times x,$$

where C is a sufficiently large compact open subset of X.

Proof: If $L\subset \mathrm{GL}(n,E)$ is the kernel of the natural map

$$\mathrm{GL}(n,\mathcal{O}_E)\to \mathrm{GL}(n,\mathcal{O}_E/\mathcal{P}_E^f),$$

then L is a compact open subgroup of $\mathrm{GL}(n,\mathcal{O}_E)$ and $\det(g)\in\ker(\tau\circ N)$ for all $g\in L$. Let φ_β be the characteristic function of the orbit $L[\beta]$ of β under L, and set $\hat{\phi}=\varphi_\beta$, so that $\phi(x)=\widehat{\varphi_\beta}(-x)$. We then apply the functional equation (3.8) to ϕ. The right-hand side yields

$$\begin{aligned}(\mathrm{RHS}) &= e_\beta(s,\tau)\int_{L[\beta]}\tau(x)^{-1}|x|_F^{-s}\,dx \\ &= e_\beta(s,\tau)\tau(\beta)^{-1}|\beta|_F^{-s}m^+(L[\beta]),\end{aligned} \tag{3a.4}$$

where $m^+(L[\beta])$ is the additive measure of $L[\beta]\subset X$. On the other hand, letting $C\supset -\mathrm{supp}(\hat{\varphi_\beta})$ be a compact open subset of X which is stable under L (this is easily chosen),

$$\begin{aligned}(\mathrm{LHS}) &= \int_C \tau(x)|x|^s\left[\int_{L[\beta]}\psi(-\mathrm{tr}(xy))\,dy\right]d^\times x \\ &= \int_{L[\beta]}\int_C \tau(x)|x|^s\bar{\psi}(\mathrm{tr}(xy))\,d^\times x\,dy.\end{aligned} \tag{3a.5}$$

Normalizing Haar measure dg on L so that $\int_L dg = 1$, the above equals

$$\begin{aligned} & m^+(L[\beta]) \int_L \int_C \tau(x)|x|^s \bar{\psi}(\mathrm{tr}(xg\beta\,{}^t\bar{g}))\, d^\times x\, dg \\ (3a.6) \qquad & = m^+(L[\beta]) \int_L \int_C \tau({}^t\bar{g}xg)|{}^t\bar{g}xg|^s \bar{\psi}(\mathrm{tr}({}^t\bar{g}xg\beta))\, d^\times x\, dg \\ & = m^+(L[\beta]) \int_C \tau(x)|x|^s \bar{\psi}(\mathrm{tr}(x\beta))\, d^\times x. \end{aligned}$$

Equating the two sides gives the result. ∎

The next lemma permits e_β to be computed by induction on n. We will write $e_\beta(s,\tau) = e_\beta^n(s,\tau)$ and $X = X^n$ to make the dependence on n apparent.

LEMMA 3A.2: *Fix a* $\mathrm{GL}(n,E)$ *orbit of* Y, *and choose a representative* β *of that orbit of the form* $\beta = \begin{pmatrix} \beta_1 & \\ & \beta_2 \end{pmatrix}$, *where* $\beta_1 \in \mathrm{diag}(n-1, F^\times)$ *and* $\beta_2 \in F^\times$. *Then for* $n \geq 2$,

$$e_\beta^n(s,\tau) = \mathcal{I}(n-1, s, \beta_1, \beta_2) \cdot e_{\beta_1}^{n-1}(s-1,\tau),$$

where

$$\mathcal{I}(n-1,s,\beta_1,\beta_2) = \tau(\beta_2)|\beta_2|_F^{\frac{s}{2}}|\beta_1|_F \int_{\mathcal{A}} \tau(c)|c|_F^{\frac{s}{2}+n-2} \bar{\psi}(\mathrm{tr}(Bc\,{}^t\bar{B}\beta_1) + c\beta_2)\, dB\, dc,$$

and the integration takes place over the set

$$\mathcal{A} = \{(B,c) \in E^{n-1} \times F \mid Bc\,{}^t\bar{B} \in \mathcal{C}_1,\ Bc \in \mathcal{C}_2, \text{ and } c \in \mathcal{C}_3\}$$

for sufficiently large additively-closed compact open sets $\mathcal{C}_i$ *in the appropriate spaces. Here,* B *is a column matrix, and the measures are as described below.*

Proof: First, note that the self-dual measure on X with respect to the character ψ is given by

$$dx = \left(\prod_{i=1}^n dx_{ii}\right) \cdot \left(\prod_{1 \leq i < j \leq n} dx_{ij}\right) \tag{3a.7}$$

where dx_{ii} is the self-dual measure on F with respect to ψ, and for $i<j$, dx_{ij} is the measure on E which is self-dual with respect to the Fourier transform on $\mathcal{S}(E)$ defined by the character $\psi \circ T$. This follows from the fact that for $x,\ y \in X$,

$$\psi(\mathrm{tr}(xy)) = \left(\prod_{i=1}^n \psi(x_{ii}y_{ii})\right) \cdot \left(\prod_{i<j} \psi(T(x_{ij}y_{ji}))\right). \tag{3a.8}$$

If an element $x \in X^n$ is written as

$$x = \begin{pmatrix} U & V \\ {}^t\bar{V} & c \end{pmatrix} \tag{3a.9}$$

for $U \in X^{n-1}$, $V \in M(n-1,1,E)$, and $c \in X^1 = F$, then we have

$$dx = dU \cdot dV \cdot dc, \tag{3a.10}$$

where dU and dc are the usual measures on X^{n-1} and X^1, respectively, and $dV = \prod_{i=1}^{n-1} dV_i$, for dV_i the measure on E defined above. As in [24] and [31], there is a homeomorphism

$$X^{n-1} \times M(n-1,1,E) \times F^\times \longrightarrow \{x \in X^n \mid x_{nn} \neq 0\} \tag{3a.11}$$

given by

$$(A,B,c) \longmapsto \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ {}^t\bar{B} & 1 \end{pmatrix} = \begin{pmatrix} A + Bc\,{}^t\bar{B} & Bc \\ c\,{}^t\bar{B} & c \end{pmatrix}. \tag{3a.12}$$

From the Lemma 3A.1,

(3a.13)

$$e^n_\beta(s,\tau) = \tau(\beta)|\beta|^s_F \int_{\mathcal{C}} \tau\begin{pmatrix} U & V \\ {}^t\bar{V} & c \end{pmatrix} \left|\begin{pmatrix} U & V \\ {}^t\bar{V} & c \end{pmatrix}\right|^{s-n}_F \bar{\psi}(\mathrm{tr}(\begin{pmatrix} U & V \\ {}^t\bar{V} & c \end{pmatrix}\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}))dU\,dV\,dc.$$

After making the change of variables $U = A + Bc\,{}^t\bar{B}$, $V = Bc$, this yields

(3a.14) $e^n_\beta(s,\tau) =$

$$\tau(\beta)|\beta|^s_F \int \tau(A)\tau(c)|A|^{s-n}_F|c|^{s-n}_F\bar{\psi}(\mathrm{tr}(A\beta_1 + Bc\,{}^t\bar{B}\beta_1) + c\beta_2)|c|^{2n-2}_F dA\,dB\,dc,$$

where we choose sets $\mathcal{C}_i$ so that (schematically) we may write

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1 & \mathcal{C}_2 \\ {}^t\bar{\mathcal{C}}_2 & \mathcal{C}_3 \end{pmatrix}, \tag{3a.15}$$

and the integral is over the set

$$\{(A,B,c) \mid A + Bc\,{}^t\bar{B} \in \mathcal{C}_1,\ Bc \in \mathcal{C}_2, \text{ and } c \in \mathcal{C}_3\}. \tag{3a.16}$$

Integrating with respect to A first, we have

$$e^n_\beta(s,\tau) = \tau(\beta)|\beta|^s_F \int_{Bc\in\mathcal{C}_2, c\in\mathcal{C}_3} \tau(c)|c|^{s+n-2}_F\bar{\psi}(\mathrm{tr}(Bc\,{}^t\bar{B}\beta_1) + c\beta_2)\times$$
$$\left[\int_{A+Bc\,{}^t\bar{B}\in\mathcal{C}_1} \tau(A)|A|^{s-1}_F\bar{\psi}(\mathrm{tr}(A\beta_1))\, d^\times A\right] dB\,dc. \tag{3a.17}$$

By the same argument used in Proposition 4.2 of [31], the integral in brackets above equals

$$\text{(3a.18)}\qquad \begin{cases} \tau(\beta_1)^{-1}|\beta_1|_F^{1-s}\cdot e_{\beta_1}^{n-1}(s-1,\beta_1) & \text{if } Bc\,{}^t\bar{B}\in\mathcal{C}_1,\\ 0 & \text{otherwise.}\end{cases}$$

Collecting terms gives the required result. ∎

To complete the induction, we still must simplify the integral $\mathcal{I}(n-1,s,\beta_1,\beta_2)$. First of all, we may choose the sets $\mathcal{C}_i$ carefully so that

(3a.19)

$$\mathcal{A}=\{(B,c)\in E^{n-1}\times F \mid c\in\mathcal{P}_F^N,\text{ and } cN(B_i)\in\mathcal{P}_E^M \text{ for } 1\le i\le n-1\}$$

for any M and N satisfying $2N\le M\ll 0$. If $\beta_1=\operatorname{diag}(\beta_1^{(1)},\dots,\beta_1^{(n-1)})$, then

$$\text{(3a.20)}\qquad \operatorname{tr}(Bc\,{}^t\bar{B}\beta_1)=\sum_{i=1}^{n-1}c\beta_1^{(i)}N(B_i),$$

and we have

$$\text{(3a.21)}\quad \mathcal{I}(n-1,s,\beta_1,\beta_2)=\tau(\beta_2)|\beta_2|_F^{s}|\beta_1|_F\times \int_{\mathcal{P}_F^N}\tau(c)|c|_F^{s+n-2}\bar{\psi}(c\beta_2)\left[\prod_{i=1}^{n-1}\int_{cN(B_i)\in\mathcal{P}_E^M}\bar{\psi}(c\beta_1^{(i)}N(B_i))\,dB_i\right]dc.$$

The integral in brackets may then be written in terms of the Weil index:

LEMMA 3A.3: *Let χ_A stand for the characteristic function of any set A. Then for $M\ll 0$,*

$$(*)\qquad \int_E \chi_{\mathcal{P}_E^M}(xN(b))\,\psi(xN(b))\,db=\gamma_E(f_x)|x|_F^{-1}$$

for any $x\in F^\times$, where $f_x\colon E^+\to\mathbb{T}$ is the character of second degree given by $f_x(y)=\psi(xN(y))$ and $\gamma_E(f_x)$ is the Weil index of f_x (see [34], [31], or [27]). The measure db on E is chosen (as above) to be self-dual with respect to $\psi\circ T$.

Proof: Fix $x\in F^\times$, and choose $L\in\mathbb{Z}$ so that $xN(b)\in\mathcal{P}_E^M$ if and only if $\operatorname{ord}(b)\ge L$ (ord standing for the order mod $\mathcal{P}_E$). Then $L=L(M,x)$ is the smallest integer which is greater than or equal to $\frac{M-\operatorname{ord}(x)}{2}$. Suppose that $\psi\circ T=1$ on $\mathcal{P}_E^{-D}$, but not on $\mathcal{P}_E^{-D-1}$, and define $A=\mathcal{P}_E^{-(L+D+\operatorname{ord}(x))}$. It is easy to see that

$$\text{(3a.22)}\qquad \int_A\psi(T(xb\bar{y}))dy=\begin{cases} m(A) & \text{if } b\in\mathcal{P}_E^L,\\ 0 & \text{otherwise.}\end{cases}$$

Hence the integral $(*)$ in the statement of the lemma can be written as

$$(3a.23)\quad (*) = \int_E \chi_{\mathcal{P}_E^L}(b)\psi(xN(b))\,db = m(A)^{-1}\int_E\int_A \psi(x[N(b)+T(b\bar{y})])\,dy\,db.$$

If $M \le -D-1$, then $\operatorname{ord}(xN(y)) \ge -D$ for all $y \in A$, and we can "complete the square":

$$(3a.24)\qquad \begin{aligned}(*) &= m(A)^{-1}\int_E\int_E \chi_A(-y)\psi(xN(b+y))\,dy\,db\\ &= m(A)^{-1}\int_E\int_E \chi_A(b-y)f_x(y)\,dy\,db.\end{aligned}$$

We are now essentially done by Corollary 2 to Theorem 2 of Weil [34], save that we must compute the modulus of the symmetric morphism $\rho_x\colon E \to \hat{E}$ associated to f_x. Since

$$(3a.25)\quad f_x(a+b)f_x(a)^{-1}f_x(b)^{-1} = \psi(x[N(a+b)-N(a)-N(b)]) = \psi(xT(a\bar{b})),$$

ρ_x takes $b \in E$ to the mapping $a \mapsto \psi(xT(a\bar{b}))$. But in our definition of the Fourier transform on E, we have already chosen an isomorphism $E \simeq \hat{E}$ via $b \mapsto \psi_b$, where $\psi_b(c) = \psi(T(bc))$, and the measure on $\hat{E}$ which is dual to our fixed measure on E is exactly the measure on $\hat{E}$ which is inherited from E via this isomorphism. Hence the modulus of ρ_x equals $|x|_E = |x|_F^2$. So finally by Weil's result cited above,

$$(3a.26)\qquad (*) = \gamma_E(f_x)|\rho_x|^{-\frac{1}{2}}m(A)^{-1}\int_E \chi_A(x)\,dx = \gamma_E(f_x)|x|_F^{-1},$$

as claimed. ∎

Next, note that by choosing a basis for E/F, the Weil index of f_x can be related to the "standard" Weil indices on F, as in Rao [27]. This allows the following simplification:

LEMMA 3A.4: *With f_x as above, and $\epsilon_{E/F}$ the unique non-trivial character of $F^\times$ which is trivial on $N(E^\times)$, we have*

$$\gamma_E(f_x) = \epsilon_{E/F}(x)\cdot\gamma_E(f_1) = \epsilon_{E/F}(x)\cdot\gamma_E(\psi\circ N)$$

for all $x \in F^\times$.

Proof: Choose $b \in F$ such that $E = F(\sqrt{b})$. If $c \in E$ is written as $c = \alpha + \beta\sqrt{b}$ for $\alpha,\ \beta \in F$, then

$$(3a.27)\qquad f_x(c) = \psi(x(\alpha^2 - b\beta^2)) = \psi(x\alpha^2)\psi(-bx\beta^2).$$

Using Rao's notation [27], in which $x\psi$ stands for the character of second degree on F defined by $x\psi(\alpha) = \psi(x\alpha^2)$, we see that

$$f_x = (x\psi) \times (-bx\psi) \tag{3a.28}$$

with respect to the isomorphism $E \simeq F \times F$ chosen above. Hence

$$\gamma_E(f_x) = \gamma_F(x\psi) \cdot \gamma_F(-bx\psi), \tag{3a.29}$$

since Weil indices respect direct products [34]. Again, in Rao's notation,

$$\begin{aligned}\frac{\gamma_E(f_x)}{\gamma_E(f_1)} &= \gamma_F(x,\psi)\gamma_F(x,-b\psi) = \gamma_F(x,\psi)^2(x,-b)_F \\ &= (x,b)_F = \epsilon_{E/F}(x)\end{aligned} \tag{3a.30}$$

by the appendix of Rao [27] (also summarized in [30]). Here, $(\,,\,)_F$ is the Hilbert symbol of F. ∎

Applying these last two lemmas to equation (3a.21), we obtain

LEMMA 3A.5:

$$\mathcal{I}(n-1,s,\beta_1,\beta_2) = \gamma_E(\bar\psi\circ N)^{n-1}\epsilon_{E/F}(\beta_1)\epsilon_{E/F}(\beta_2)^{n-1}\rho_F(s,\tau\cdot\epsilon_{E/F}^{n-1}).$$

Proof:

(3a.31)

$$\begin{aligned}&\mathcal{I}(n-1,s,\beta_1,\beta_2)\\ &=\tau(\beta_2)|\beta_2|_F^s|\beta_1|_F\\ &\quad\times\int_{\mathcal{P}_F^N}\tau(c)|c|_F^{s+n-2}\bar\psi(c\beta_2)\left[\prod_{i=1}^{n-1}\epsilon_{E/F}(c\beta_1^{(i)})\,\gamma_E(\bar\psi\circ N)|c\beta_1^{(i)}|_F^{-1}\right]dc\\ &=\gamma_E(\bar\psi\circ N)^{n-1}\int_{\mathcal{P}_F^N}\tau(c\beta_2)|c\beta_2|_F^s\bar\psi(c\beta_2)\,\epsilon_{E/F}(c^{n-1}\det(\beta_1))\,d^\times c\\ &=\gamma_E(\bar\psi\circ N)^{n-1}\epsilon_{E/F}(\beta_1)\epsilon_{E/F}(\beta_2)^{n-1}\int_{\mathcal{P}_F^{N'}}(\tau\cdot\epsilon_{E/F}^{n-1})(x)|x|_F^s\bar\psi(x)\,d^\times x.\end{aligned}$$

Now, applying the functional equation (3a.2) to $\varphi(x) = \bar\psi(x)\cdot\chi_{\mathcal{P}_F^N}$, it is easy to see that $\zeta(\hat\varphi,\omega) = 1$ for any quasi-character ω (if $N \ll 0$ is small enough), so that the result follows. ∎

An easy induction on n gives the final result:

PROPOSITION 3A.6: *For any $\beta\in Y$, and any character τ of $F^\times$, we have*

$$e_\beta(s,\tau) = \gamma_E(\bar\psi\circ N)^{\frac{(n-1)n}{2}}\epsilon_{E/F}(\beta)^{n-1}\prod_{r=0}^{n-1}\rho_F(s+r-(n-1),\tau\cdot\epsilon_{E/F}^r).$$

4. Submodules

Now suppose that (s_0, χ) is a possible point of reducibility of $I(s,\chi)$ as given by Theorem 3.6. In this section, we will use the Weil representation to produce proper G-submodules for most of these points. Since $\chi = \check{\chi}$, write $\chi|_{F^\times} = \epsilon_{E/F}^m$ for any integer $m \geq 0$ of the correct parity. Notice that if $s_0 \neq 0$, then $s_0 \in \mathbb{Z}$ if and only if m and n have the same parity; otherwise $s_0 \in (\frac{1}{2} + \mathbb{Z})$. So we let $s_0 = \frac{m-n}{2}$, and note that this accounts for all possible points of reducibility, save when $s_0 = 0$ and the parity of m is different from that of n. In this last case, we will prove, in a later section, that $I(0,\chi)$ is irreducible.

For each $m \geq 1$, let V, $(\,,)$ be a non-degenerate Hermitian vector space over E with $\dim_E(V) = m$. For a given m, there are exactly two such spaces (up to isometry), distinguished by the two possibilities for $\det((v_i, v_j)) \in F^\times/N(E^\times)$, where $\{v_i\}_{i=1}^m$ is some choice of an E-basis for V. Let $W = E^{2n}$ be the space of row vectors with skew-Hermitian form given by $\langle u, v\rangle = uJ\,{}^t\bar{v}$, so that $G = U(W)$ is the isometry group of W. Then we can construct a non-degenerate skew-symmetric F-linear form (a symplectic form) on the F-vector space $\mathbf{W} = V \otimes_E W$ via

$$\ll , \gg = T_F^E((\,,) \otimes \overline{\langle\,,\rangle}), \tag{4.1}$$

and, in the usual way, this gives a dual reductive pair

$$U(V) \times G \hookrightarrow \mathrm{Sp}(\mathbf{W}). \tag{4.2}$$

If $\Pi\colon \widetilde{\mathrm{Sp}(\mathbf{W})} \to \mathrm{Sp}(\mathbf{W})$ is the metaplectic extension cover of $\mathrm{Sp}(\mathbf{W})$, then ([15]) gives an explicit splitting of the extension $\Pi^{-1}(G)$ of G. This depends on the choice of a character extending $\epsilon_{E/F}^m\colon F^\times \to \mathbb{T}$ to $E^\times$, and we choose the character χ in our situation. When the Weil representation Ω_ψ of $\widetilde{\mathrm{Sp}(\mathbf{W})}$, associated to ψ, is realized on the space $\mathcal{S}(V^n)$, as is usual when restricting to dual pairs, and when we compose Ω_ψ with the splitting just mentioned, we obtain a representation $\omega = \omega_\psi$ of $U(V) \times G$ in the space $\mathcal{S}(V^n)$.

LEMMA 4.1: *The representation ω described above is given by the following formulas.*

(1) *For $G = U(n,n)$, if $\varphi \in \mathcal{S}(V^n)$, $a \in \mathrm{GL}(n,E)$, $b = {}^t\bar{b} \in M(n,E)$, and w_r is*

as in the proof of Lemma 2.1, then

$$\omega(m(a))\varphi(x) = \chi(\det(a))|\det(a)|_E^{\frac{m}{2}}\,\varphi(x\cdot a),$$
$$\omega(n(b))\varphi(x) = \psi(\mathrm{tr}((x,x)b))\,\varphi(x),$$

and

$$\omega(w_r)\varphi(x) = \gamma^{-r}\int_{V^r}\psi(T_F^E\circ\mathrm{tr}(x'',z))\,\varphi(x'+z)\,dz.$$

The representation ω is defined on all of G by the Bruhat decomposition: if $g = p_1 w_r p_2$ for $p_1,\ p_2 \in P$, then $\omega(g)$ is defined by

$$\omega(g) = \omega(p_1)\omega(w_r)\omega(p_2).$$

The notation is as follows: For $1 \le r \le n$, write $V^n = V^{n-r}\oplus V^r$ and $x = x' + x''$ to indicate that $x' \in V^{n-r}$, and $x'' \in V^r$. The measure dz on V^r is chosen to be the product of the Haar measures on V which are self-dual with respect to the Fourier transform defined by the pairing $\psi\circ T_F^E(\,,\,)\colon V\times V\to\mathbb{T}$. γ is a certain 8^{th} root of unity depending on V and χ which is given explicitly in [15].

(2) *For $U(V)$, ω is given by $\omega(h)\varphi(x) = \varphi(h^{-1}x)$, where $x\in V^n$, and*

$$h^{-1}x = h^{-1}(x_1,\dots,x_n) = (h^{-1}x_1,\dots,h^{-1}x_n).$$

A parallel construction for a symplectic-orthogonal dual pair is given in [30], for example, or in many other sources.

For any character χ restricting to $\epsilon_{E/F}^m$, and any choice of V as above, we have a G-intertwining map

$$\mathcal{S}(V^n)\longrightarrow I(s_0,\chi) \tag{4.3}$$

given by

$$\varphi\longmapsto\{g\mapsto\omega(g)\varphi(0)\}. \tag{4.4}$$

Denote the image of this map by $R_n(V,\chi)$. Rallis's Theorem on coinvariants [25] is extended to the unitary case in [23], so that

$$R_n(V,\chi)\simeq\mathcal{S}(V^n)_{U(V)}, \tag{4.5}$$

where $\mathcal{S}(V^n)_{U(V)}$ is the maximal quotient of $\mathcal{S}(V^n)$ on which $U(V)$ acts trivially.

The analogue of Proposition 3.1 of [19] holds:

PROPOSITION 4.2: *Assume that* $1 \le m = \dim_E(V) \le n$, *so that* $s_0 \le 0$. *Then* $R_n(V,\chi)$ *is an irreducible and unitarizable* G*-module, and the restriction of this representation to* P *is also irreducible.*

The proof follows that of [19] closely. In the next proposition, we begin to gather together the facts about the constituents at the points of reducibility. One definition is needed first. As in [27] and [15], we define a map

$$x\colon G \to E^\times/N(E^\times) \tag{4.6}$$

via the relative Bruhat decomposition of G with respect to P. For example, on the open cell

$$x(n_1 m(a_1) w_n n_2 m(a_2)) = \det(a_1)\det(a_2) \bmod N(E^\times) \tag{4.7}$$

for any $n_1, n_2 \in N$, and $a_1, a_2 \in \mathrm{GL}(n,E)$. For the definition of x on the other cells, we refer the reader to [15].

PROPOSITION 4.3:

(1) *Suppose that* $1 \le m \le n$ *and* $\chi|_{F^\times} = \epsilon_{E/F}^m$, *so that* $s_0 = \frac{m-n}{2} \le 0$. *Let* V_1 *and* V_2 *be inequivalent* m*-dimensional Hermitian spaces over* E. *Then the spaces* $R_n(V_1,\chi)$ *and* $R_n(V_2,\chi)$ *are inequivalent, irreducible* G*-submodules of* $I(s_0,\chi)$. *In fact, we have*

$$R_n(V_1,\chi) \oplus R_n(V_2,\chi) \subset I(s_0,\chi),$$

and these are the unique irreducible G*-submodules. In the case* $s_0 = 0$, *we have equality:*

$$R_n(V_1,\chi) \oplus R_n(V_2,\chi) = I(s_0,\chi).$$

(2) *If* $s_0 = -\frac{n}{2}$, *suppose that* $\chi|_{F^\times} = 1$, *and define* $\chi_G\colon G \to \mathbb{T}$ *by* $\chi_G(g) = \chi(x(g))$. *Then* χ_G *is a character of* G *extending* $\chi \circ \det$ *on* M, *and if we define*

$$R_n(0,\chi) = \mathbb{C}\cdot\chi_G,$$

then $R_n(0,\chi)$ *is the unique irreducible* G*-submodule of* $I(s_0,\chi)$. *(Notice that if* $s_0 = -\frac{n}{2}$ *and* $\chi|_{F^\times} = \epsilon_{E/F}$, *then* $I(s_0,\chi)$ *is irreducible by Theorem 3.6. In this case,* χ_G *is not a character.)*

(3) *If V is a Hermitian space with $\dim_E(V) = m > 2n$, or if $m = 2n$ and V is split, then*

$$I(s_0, \chi) = R_n(V, \chi).$$

In discussing the proof of this result, we summarize the details of the analogous work in [19]. As before, let $X = \{x \in M(n, E) \mid x = {}^t\bar{x}\}$, and for a Hermitian space $\{V, (\,,\,)\}$, define the moment map

$$\mu\colon V^n \longrightarrow X \tag{4.8}$$

by

$$\mu(x) = ((x_i, x_j)), \qquad \text{where } x = (x_1, \ldots, x_n) \in V^n. \tag{4.9}$$

For any $\beta \in X$, let ψ_β be the character of $N \simeq X$ given by $\psi_\beta(n(b)) = \psi(\mathrm{tr}(b\beta))$. We define a special subset of V^n:

$$V^n_{\mathrm{reg}} = \{x \in V^n \mid \text{the } E\text{-ranks of } x \text{ and } \mu(x) \text{ equal } \min\{m, n\}\}. \tag{4.10}$$

Here, the E-rank of x is the dimension of the E-subspace of V spanned by the components of x, while the rank of $\mu(x)$ is its rank as a matrix in $M(n, E)$. By a basic calculation ([1], [25], [23]), we have:

LEMMA 4.4: *Let $\beta \in X$.*

(1) *The twisted Jacquet functor $\mathcal{S}(V^n) \to \mathcal{S}(V^n)_{N,\psi_\beta}$ can be explicitly realized by the restriction map $\mathcal{S}(V^n) \to \mathcal{S}(\mu^{-1}(\beta))$.*
(2) *In particular, if $\mu^{-1}(\beta) = \emptyset$, then $\mathcal{S}(V^n)_{N,\psi_\beta} = 0$.*
(3) *If $\beta \in \mu(V^n_{\mathrm{reg}})$, then $\mu^{-1}(\beta)$ is a single $U(V)$-orbit, and the space*

$$R_n(V,\chi)_{N,\psi_\beta} \simeq \big(\mathcal{S}(V^n)_{U(V)}\big)_{N,\psi_\beta} \simeq \big(\mathcal{S}(V^n)_{N,\psi_\beta}\big)_{U(V)}$$

is one-dimensional. The map $\mathcal{S}(V^n) \to R_n(V,\chi)_{N,\psi_\beta}$ is given by integration against a $U(V)$-invariant measure on $\mu^{-1}(\beta)$.

Proof of Proposition 4.3: (1) If V_1, $(\,,\,)_1$ and V_2, $(\,,\,)_2$ are inequivalent non-degenerate Hermitian spaces of dimension m with $m \le n$, then we have two moment maps:

$$\mu_r\colon V_r^n \longrightarrow X, \qquad r = 1,\ 2. \tag{4.11}$$

Choosing bases $\{v_1^{(r)}, v_2^{(r)}, \ldots, v_m^{(r)}\}$ for each V_r, let Q_r be the matrix of scalar products $((v_i^{(r)}, v_j^{(r)}))_r$ for $r = 1$ and 2, respectively. That the spaces V_r are inequivalent amounts exactly to saying that Q_1 and Q_2 are not equivalent under the usual $\mathrm{GL}(m, E)$ action on $H(m, E)$. With the isomorphisms $V_r^n \simeq M(m,n,E)$ induced by our choice of bases, we clearly have $\mu_r(x) = {}^t x Q_r \bar{x}$ for $x \in M(m,n,E)$, and it is not hard to show that

$$\mu_1(V^n_{1,\,\mathrm{reg}}) = \mathrm{GL}(n,E) \cdot \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.12}$$

and similarly for V_2. But then the failure of Q_1 and Q_2 to be $\mathrm{GL}(m,E)$ equivalent means that in fact

$$\mu_1(V^n_{1,\,\mathrm{reg}}) \cap \mu_2(V^n_{2,\,\mathrm{reg}}) = \emptyset. \tag{4.13}$$

Using Lemma 4.4 above, we can then show that $R_n(V_1,\chi) \not\simeq R_n(V_2,\chi)$. Fix $\beta_1 = \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix}$, and consider the following. Applying the exact Jacquet functor $(\ )_{N,\psi_{\beta_1}}$ to the exact sequence

$$0 \to \mathcal{S}(V_2^n)(U(V_2)) \to \mathcal{S}(V_2^n) \to R_n(V_2,\chi) \to 0, \tag{4.14}$$

yields another exact sequence

$$0 \to \mathcal{S}(V_2^n)(U(V_2))_{N,\psi_{\beta_1}} \to \mathcal{S}(V_2^n)_{N,\psi_{\beta_1}} \to R_n(V_2,\chi)_{N,\psi_{\beta_1}} \to 0. \tag{4.15}$$

But now $\mu_2^{-1}(\beta_1) = \emptyset$ by our observation above, and so by the lemma,

$$R_n(V_2,\chi)_{N,\psi_{\beta_1}} = 0. \tag{4.16}$$

On the other hand, $\beta_1 \in \mu_1(V^n_{1,\,\mathrm{reg}})$ implies that $R_n(V_1,\chi)_{N,\psi_{\beta_1}}$ is one-dimensional. Hence the spaces $R_n(V_r,\chi)$ cannot be isomorphic as G-modules, and the first part of (1) follows by Propositions 2.3 and 4.2. The statement about $I(0,\chi)$ is due to the fact that $I(0,\chi)$ is completely reducible, taken together with Proposition 2.3.

(2) By Proposition 2.3 again, the main point which must be checked is that χ_G is a character. But this follows from the fact that for any $g_1, g_2 \in G$, the quotient

$$\frac{x(g_1g_2)}{x(g_1)x(g_2)} \tag{4.17}$$

actually lies in $F^\times$, although the factors are in $E^\times$ ([15]).

(3) Suppose that $m \geq n$, and define

$$V^n_{\mathrm{sub}} = \{x \in V^n \mid \text{the } E\text{-rank of } x \text{ is } n\}. \tag{4.18}$$

Exactly as in [19], under the assumption that the moment map

$$\mu\colon V^n_{\mathrm{sub}} \longrightarrow X \tag{4.19}$$

is surjective, the Weil orbital integral mapping [35]

$$\begin{aligned} \mathcal{S}(V^n_{\mathrm{sub}}) &\longrightarrow \mathcal{S}(X) \\ \varphi &\longmapsto M_\varphi \end{aligned} \tag{4.20}$$

will be surjective also. For $b \in X$, we will then have

$$\begin{aligned} \omega(w_n n(b))\varphi(0) &= \gamma^{-n} \int_{V^n} \psi(\mathrm{tr}(b\mu(x))\varphi(x)\,dx \\ &= \gamma^{-n} \int_X \psi(\mathrm{tr}(by))M_\varphi(y)\,dy \\ &= \gamma^{-n}\hat{M}_\varphi(b). \end{aligned} \tag{4.21}$$

So if our assumption on μ holds, then the space

$$J^n = \{\Phi \in I(s_0,\chi) \mid \mathrm{supp}(\Phi) \subset Pw_nN\}$$

will be spanned by $R_n(V,\chi)$, and since J^n generates $I(s_0,\chi)$ as a G-module, we will have $I(s_0,\chi) = R_n(V,\chi)$. But (4.20) is surjective if and only if V has isotropic subspaces of dimension n, and this only occurs if $m > 2n$, or if $m = 2n$ and V is a split form (this is easily shown by actually constructing the two possible forms of dimension m). ∎

5. Jacquet modules again

If we choose the standard basis $\{e_1,\ldots,e_n,e_1^*,\ldots,e_n^*\}$ for $W = E^{2n}$, and realize $G \subset \mathrm{GL}(2n,E)$, as before, we may define a maximal parabolic subgroup $P_1 \subset G$ to be the stabilizer of the subspace $E\cdot e_1^*$. Then $P_1 = M_1N_1$, where

(5.1)

$$M_1 = \left\{ \begin{pmatrix} a & & 0 & \\ & \alpha & & \beta \\ 0 & & \bar{a}^{-1} & \\ & \gamma & & \delta \end{pmatrix} \in G \mid a \in \mathrm{GL}(1,E) \text{ and } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(n-1,n-1) \right\},$$

$$N_1 = \left\{ m\begin{pmatrix} 1 & b \\ 0 & 1_{n-1} \end{pmatrix} n\begin{pmatrix} c & d \\ {}^t\bar{d} & 0 \end{pmatrix} \mid b,d \in E^{n-1} \text{ and } c \in F \right\}.$$

For the remainder of the paper, we will use a subscript n to denote objects relating to $G = G_n = U(n,n)$: for example, $I_n(s,\chi)$, $R_n(V,\chi)$ and $M_n^*(s,\chi)$ are all G_n-modules or morphisms, respectively. By computing the Jacquet module of $I_n(s,\chi)$ with respect to N_1, we obtain a relationship between $I_n(\,\cdot\,,\chi)$ and $I_{n-1}(\,\cdot\,,\chi)$ which will be useful for inductive arguments.

PROPOSITION 5.1: *There is an exact sequence of $M_1 \simeq \mathrm{GL}(1,E)\times G_{n-1}$ modules as follows:*

$$0 \longrightarrow \check{\chi}|\,|^{-s+\frac{n}{2}} \otimes I_{n-1}(s-\frac{1}{2},\chi) \xrightarrow{\alpha} I_n(s,\chi)_{N_1} \xrightarrow{\beta} \chi|\,|^{s+\frac{n}{2}} \otimes I_{n-1}(s+\frac{1}{2},\chi) \longrightarrow 0.$$

This sequence splits when $\check{\chi}|\,|^{-s} \neq \chi|\,|^{s}$. If (s,χ) is normalized, this occurs when $s \neq 0$ or $\chi \neq \check{\chi}$. Here β is induced by restriction to M_1, while α is described in the proof of Lemma 5.5 below.

The proof follows that in [19] closely. For later use, we record the following lemma concerning the various possible Hermitian spaces of a given dimension.

LEMMA 5.2: *Let V be a non-degenerate Hermitian vector space over E with dimension m. Fix some element $\alpha \in F^\times \sim N_F^E(E^\times)$.*

(1) *If $m = 1$, then there are two possibilities for the isometry class of V, given by $V_i = E$ and $\langle x,y\rangle_{Q_i} = xQ_i\bar{y}$, where $Q_1 = 1$ and $Q_2 = \alpha$. Both of these are clearly anisotropic.*

(2) *If $m = 2$, then the two possibilities are given by $V_i = E^2$ (row vectors), and $\langle x,y\rangle_{Q_i} = xQ_i\,{}^t\bar{y}$, where*

$$Q_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \sim \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} \alpha & \\ & -1 \end{pmatrix}.$$

Here, V_1 is a hyperbolic plane, and V_2 is anisotropic.

(3) *If $m \geq 3$, then V consists of the direct sum of a certain number of hyperbolic planes with an anisotropic kernel which has dimension 0, 1, or 2, and which appears above. In particular, when $m \geq 3$, each of the possibilities for V has a non-zero isotropic vector.*

Now we restrict the exact sequence from Proposition 5.1 to the Jacquet module $R_n(V,\chi)_{N_1}$ at the special value $s_0 = \frac{m-n}{2}$.

PROPOSITION 5.3: *Suppose that* $\chi|_{F^\times} = \epsilon_{E/F}^m$, $s_0 = \frac{m-n}{2}$, *and let* V *be a Hermitian space with dimension* m. *If* V *is isotropic, let* V' *be the space derived from* V *by removing a hyperbolic plane. Then*

(1) *As* $M_1 \simeq \mathrm{GL}(1,E) \times G_{n-1}$ *modules, the sequence*

$$0 \longrightarrow \chi|\,|^{n-\frac{m}{2}} \otimes I_{n-1}(s_0 - \tfrac{1}{2}, \chi) \xrightarrow{\alpha} I_n(s_0, \chi)_{N_1} \xrightarrow{\beta} \chi|\,|^{\frac{m}{2}} \otimes I_{n-1}(s_0 + \tfrac{1}{2}, \chi) \longrightarrow 0$$

is exact.

(2) *As* $M_1 \simeq \mathrm{GL}(1,E) \times G_{n-1}$ *modules, the sequence*

$$0 \longrightarrow \chi|\,|^{n-\frac{m}{2}} \otimes R_{n-1}(V', \chi) \xrightarrow{\alpha'} R_n(V,\chi)_{N_1} \xrightarrow{\beta'} \chi|\,|^{\frac{m}{2}} \otimes R_{n-1}(V,\chi) \longrightarrow 0$$

is exact. When V *is anisotropic,* $R_{n-1}(V',\chi)$ *is taken to be zero.*

(3) *The natural maps between terms of the two sequences yield a commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \to & I_{n-1}(s_0 - \frac{1}{2}, \chi) & \to & I_n(s_0,\chi)_{N_1} & \to & I_{n-1}(s_0 + \frac{1}{2}, \chi) & \to & 0 \\ & & i'' \uparrow & & i \uparrow & & \uparrow & & \\ 0 & \to & R_{n-1}(V',\chi) & \to & R_n(V,\chi)_{N_1} & \to & R_{n-1}(V,\chi) & \to & 0. \end{array}$$

where i *is the natural map, and* i'' *is a non-zero multiple of the natural map, and, to save space, the character* $\chi|\,|^{n-\frac{m}{2}}$ *(resp.* $\chi|\,|^{\frac{m}{2}}$*) has been omitted from the leftmost (resp. rightmost) terms.*

Again, the proof is as in [19]. Next, we use these last results to prove irreducibility at the remaining series of points on the unitary axis.

PROPOSITION 5.4: *Suppose that* $\chi|_{F^\times} = \epsilon_{E/F}^m$, *and that* m *and* n *have different parity. Then* $I(0,\chi)$ *is irreducible.*

Note that this Proposition, together with Theorem 3.6 and the accumulated information on submodules of $I_n(s_0,\chi)$, completes the proof of Theorem 1.1.

Proof: Since $I(0,\chi)$ is completely reducible and has no more than two irreducible submodules, by Proposition 2.3, it is easy to see that if $I(0,\chi)$ were reducible, then it would be the direct sum of two irreducible submodules. Using the faithfulness of the N_1 Jacquet functor, one could then show that the exact sequence of Proposition 5.1 would split. But this does not happen in our situation, in fact, by the next lemma. ∎

LEMMA 5.5: *If $\chi|_{F^\times} = \epsilon_{E/F}^m$ and m, n have different parity, then the sequence*

$$0 \longrightarrow \chi|\,|^{\frac{n}{2}} \otimes I_{n-1}(-\frac{1}{2},\chi) \xrightarrow{\alpha} I_n(0,\chi)_{N_1} \xrightarrow{\beta} \chi|\,|^{\frac{n}{2}} \otimes I_{n-1}(\frac{1}{2},\chi) \longrightarrow 0$$

does not split.

Proof: It suffices to prove that $I_n(0,\chi)_{N_1}$ does not transform by the character $\chi|\,|^{\frac{n}{2}}$ under the action of the copy of GL(1) in M_1. For any $\Phi \in I_n(0,\chi)_{N_1}$ and $t \in \mathrm{GL}(1)$ we write $r(t)$ for the action of t on Φ. It is clear that $\beta(r(t)\Phi - \chi(t)|t|^{\frac{n}{2}}\Phi) = 0$, so that $r(t)\Phi - \chi(t)|t|^{\frac{n}{2}}\Phi$ lies in $\ker\beta$, and we may apply α^{-1} to it. We claim that $\alpha^{-1}(r(t)\Phi - \chi(t)|t|^{\frac{n}{2}}\Phi) \neq 0$ for some Φ, which will suffice to prove the lemma.

As in [19], we consider the mapping used to define α^{-1} on $\ker\beta$. Since β is given by restriction to M_1, and $G = P_nP_1 \coprod P_nwP_1$, where

$$w = \begin{pmatrix} 0 & & 1 & \\ & 1_{n-1} & & 0 \\ -1 & & 0 & \\ & 0 & & 1_{n-1} \end{pmatrix}, \tag{5.2}$$

$\ker\beta$ is just the image in $I_n(0,\chi)_{N_1}$ of the space of functions

$$T = \{\Psi \in I_n(0,\chi) \mid \mathrm{supp}(\Phi) \subset P_nwP_1\}. \tag{5.3}$$

Then α^{-1} is defined on the image of T by the integral operator

$$(A\Phi)(g) = \int_{N_1'} \Phi(wug)\,du, \tag{5.4}$$

where

$$N_1' = \left\{ m\begin{pmatrix} 1 & x \\ 0 & 1_{n-1} \end{pmatrix} n\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \mid x \in E^{n-1},\ y \in F \right\}, \tag{5.5}$$

and $g \in M_1$. This converges if $\Phi \in T$, since the integrand has compact support: consider the maps

$$\begin{aligned} P_1 \overset{\pi_1}{\to} P_n\backslash P_nwP_1 &\overset{\pi_2}{\leftarrow} P_nwP_1 \\ p \mapsto P_nwp&, \end{aligned} \tag{5.6}$$

and where π_2 is the natural projection. Since N_1' is homeomorphic to $\pi_1(N_1')$, if we let $f(u) = \Phi(wu)$ for $u \in N_1'$ and $\Phi \in T$, then $\mathrm{supp}(f) \simeq \pi_1(\mathrm{supp}(f)) =$

$\pi_2(\text{supp}(\Phi)) \cap \pi_1(N_1')$. This last is the intersection of a compact set with a closed set, and hence f has compact support.

If the support of a standard section $\Phi \in I_n(s,\chi)$ is arbitrary, then the integral

$$(5.4) \qquad \big(A(s)(\Phi(s))(g) = \int_{N_1'} \Phi(wug, s)\, du,$$

converges for large $\text{Re}(s)$. The integral operator $A(s)$ has a meromorphic continuation to the whole s-plane. We will show in a moment that $A(s)$ has at most a simple pole at $s=0$, and we write

$$A(s) = \frac{A_{-1}}{s} + A_0 + O(s)$$

for its Laurent expansion at this point. Also, a simple computation shows that, for $g_1 \in M_1$ and for $\text{Re}(s)$ large,

$$A(s)\big(r(t)\Phi(s)\big)(g_1) = \chi(t)|t|_E^{-s+\frac{n}{2}} A(s)\big(\Phi(s)\big)(g_1).$$

Choose a standard section $\Phi(s) \in I_n(s,\chi)$ and $g_1 \in M_1$, so that $A(s)(\Phi(s))(g_1)$ has a non-zero residue at $s=0$, i.e., such that $A_{-1}(\Phi)(g_1) \neq 0$. Note that, for all s, the function $r(t)\Phi(s) - \chi(t)|t_E^{s+\frac{n}{2}}\Phi(s)$ has support in $P_n w P_1$, and thus

$$A(s)\big(r(t)\Phi(s) - \chi(t)|t_E^{s+\frac{n}{2}}\Phi(s)\big)(g_1)$$

is entire and its value at $s=0$ is simply

$$A\big(r(t)\Phi(0) - \chi(t)|t_E^{\frac{n}{2}}\Phi(0)\big)(g_1).$$

On the other hand, for $\text{Re}(s)$ large,

$$\begin{aligned}
&A(s)\big(r(t)\Phi(s) - \chi(t)|t_E^{s+\frac{n}{2}}\Phi(s)\big)(g_1)\\
=&A(s)\big(r(t)\Phi(s)\big)(g_1) - \chi(t)|t_E^{s+\frac{n}{2}} A(s)\big(\Phi(s)\big)(g_1)\\
=&\chi(t)|t|_E^{\frac{n}{2}}\big(|t|_E^{-s} - |t|_E^{s}\big)A(s)\big(\Phi(s)\big)(g_1).
\end{aligned}$$

Since the expression $|t|_E^{-s} - |t|_E^{s}$ has a simple zero at $s=0$, while $A(s)\big(\Phi(s)\big)(g_1)$ has a simple pole there wth non-zero residue, we see that

$$A\big(r(t)\Phi(0) - \chi(t)|t_E^{\frac{n}{2}}\Phi(0)\big)(g_1) \neq 0,$$

and hence that GL(1) does not act by the character $t \mapsto \chi(t)|t|^{\frac{n}{2}}$ in $I_n(0,\chi)_{N_1}$.

It remains to show that $A(s)$ has a simple pole at $s=0$. Defining $i\colon G_{n-1} \to G_n$ via the isomorphism and inclusion $\mathrm{GL}(1) \times G_{n-1} \simeq M_1 \subset G_n$, one then computes that

$$M_{n-1}(s-\tfrac{1}{2},\chi) \circ A(s) = i^* \circ M_n(s,\chi). \tag{5.7}$$

But in our current situation, $M_n(s,\chi)$ has a simple pole at $s=0$: this is the pole which was removed by renormalizing via division by

$$a(s,\chi) = \prod_{j=0}^{n-1} \zeta_F(2s+j-(n-1), \chi \cdot \epsilon_{E/F}^j).$$

Hence $(M_n(s,\chi)\Phi)(e,s)$ has a simple pole at 0 for some Φ. But one also checks that $M_{n-1}(s-\frac{1}{2},\chi)$ is holomorphic and injective at $s=0$ (see Proposition 3.2), so that $A(s)$ has precisely a simple pole there. ∎

PROPOSITION 5.6: *Let $s_0 = \frac{m-n}{2}$ with $m \geq n$ and $\chi|_{F^\times} = \epsilon_{E/F}^m$. If V_1 and V_2 are inequivalent m-dimensional Hermitian vector spaces over E, then*

$$I_n(s_0,\chi) = R_n(V_1,\chi) \dotplus R_n(V_2,\chi).$$

The proof follows that of Proposition 5.3 of [19], with the obvious modifications.

If V is a Hermitian space of dimension m with $n \leq m \leq 2n$, then let V_0 be the Hermitian space of dimension $2n-m$ such that $V \oplus -V_0$ is the split space of dimension $2n$. Here, $-V_0$ denotes the space V_0 with inner product equal to the negative of the original. The space V_0 exists in all cases except when V is the non-split space of dimension $2n$. When V is the split space of dimension $2n$, we take $V_0=0$. Notice that $s_0 = \frac{m-n}{2}$ implies that $-s_0 = \frac{(2n-m)-n}{2}$, and also that m and $2n-m$ have the same parity, so that $R_n(V,\chi) \subset I_n(s_0,\chi)$ and $R_n(V_0,\chi) \subset I_n(-s_0,\chi)$ can be constructed with the same character χ, in the case that $\chi|_{F^\times} = \epsilon_{E/F}^m$.

Now consider $\Phi_1 \in I_n(s,\chi)$ and $\Phi_2 \in I_n(-\bar{s},\chi)$. The function $\varphi(g) = \Phi_1(g) \cdot \overline{\Phi_2(g)}$ satisfies $\varphi(n(b)m(a)g) = |a|^n \varphi(g)$, so that φ lies in a space of functions which carries a unique right G-invariant Haar measure (see [1], p.11). In fact, this measure is realized by the formula $\varphi \mapsto \int_X \varphi(w_n n(b))\,db$, so that we have a conjugate-linear pairing on $I_n(s,\chi) \times I_n(-\bar{s},\chi)$ defined by

$$\langle \Phi_1, \Phi_2 \rangle = \int_X \Phi_1(w_n n(b), s)\overline{\Phi_2(w_n n(b), -\bar{s})}\,db. \tag{5.8}$$

Now suppose that $n \leq m \leq 2n$, $s_0 = \frac{m-n}{2}$, $\chi|_{F^\times} = \epsilon^m_{E/F}$, and we have a Hermitian space V of dimension m with complementary space V_0. If $\Phi_1 \in R_n(V,\chi)$ is the image of $\varphi_1 \in \mathcal{S}(V^n)$, and similarly for $\Phi_2 \in R_n(V_0,\chi)$ and $\varphi_2 \in \mathcal{S}(V_0^n)$, then for some non-zero constant c,

$$(5.9) \quad \langle \Phi_1, \Phi_2 \rangle = c\int_X \int_{V^n} \int_{V_0^n} \psi(\operatorname{tr} b[(x,x)_V - (y,y)_{V_0}])\varphi_1(x)\bar{\varphi}_2(y)\, dx\, dy\, db.$$

Letting $W = V \oplus (-V_0)$ as above, and writing $\varphi = \varphi_1 \otimes \bar{\varphi}_2 \in \mathcal{S}(W^n)$, up to a constant, the above pairing equals

$$(5.10) \quad \begin{aligned} \int_X \int_{W^n} \psi(\operatorname{tr} b(z,z)_W)\varphi(z)\, dz\, db &= \int_X \int_X \psi(\operatorname{tr} by) M_\varphi(y)\, dy\, db \\ &= \hat{\hat{M}}_\varphi(0) = M_\varphi(0), \end{aligned}$$

where M_φ is the Weil orbital integral map discussed in the proof of Proposition 4.3. In fact, if the moment mapping μ: $W^n_{\text{sub}} \to X$ is given as usual by $\mu(x) = (x,x)_W$, then M_φ is given by the integral formula

$$(5.11) \quad M_\varphi(y) = \int \varphi(x) d\nu_y(x),$$

where, for each $y \in X$, ν_y is a measure on the space $\mu^{-1}(y)$. Now, $W^n_{\text{sub}} = \{x \in W^n \mid \operatorname{rank}(x) = n\}$, and so $\mu^{-1}(0) \neq \emptyset$ if and only if W has isotropic subspaces of dimension n, which is the case by our construction of W. So we may choose φ so that the pairing $\langle \Phi_1, \Phi_2 \rangle$ is non-zero for $\Phi_1 \in R(V,\chi)$, $\Phi_2 \in R(V_0,\chi)$. Similarly, it is clear that if U is another Hermitian space of dimension $2n - m$ which is not complementary to V, then $R(V,\chi)$ is orthogonal to $R(U,\chi)$.

Now define $\overline{R(V_0,\chi)}$ to be the space of vectors $\bar{v}$, where $v \in R(V_0,\chi)$, and with $\alpha \in \mathbb{C}$ acting via $(\alpha, \bar{v}) \mapsto \alpha \cdot \bar{v}$, and $g \in G$ acting via $(g,\bar{v}) \mapsto \overline{g \cdot v}$. It is easily checked that the pairing above defines a non-zero element of $\operatorname{Hom}_G(R(V,\chi), \widetilde{\overline{R(V_0,\chi)}})$ by setting $v_1(\bar{v}_2) = \langle v_1, v_2 \rangle$ for $v_1 \in R(V,\chi)$ and $v_2 \in R(V_0,\chi)$.

Note that the whole picture above works out perfectly well when V is the split space of dimension $2n$, using $V_0 = 0$, and defining $R(0,\chi) = \mathbb{C} \cdot \chi_G$ as in Proposition 4.3.

PROPOSITION 5.7: *If V is an m-dimensional Hermitian space with $n \leq m \leq 2n$ which has a complementary space V_0, then*

$$\operatorname*{Hom}_G(R(V,\chi), \overline{R(V_0,\chi)}^{\sim}) \neq 0.$$

In fact, since $R(V_0,\chi)$ is unitarizable, we have $\overline{R(V_0,\chi)}^{\sim} \simeq R(V_0,\chi)$, and so

$$\operatorname*{Hom}_G(R(V,\chi), R(V_0,\chi)) \neq 0.$$

PROPOSITION 5.8: *Suppose that $n \leq m < 2n$, so that if V_1 and V_2 are the two inequivalent Hermitian spaces of dimension m, the complementary spaces $V_{1,0}$ and $V_{2,0}$ both exist. Suppose also that $\chi|_{F^\times} = \epsilon_{E/F}^m$. Then for $i = 1,2$,*

$$M^*(s_0,\chi)(R_n(V_i,\chi)) = R_n(V_{i,0},\chi)$$

and so

$$M^*(s_0,\chi)(I_n(s_0,\chi)) = R_n(V_{1,0},\chi) \oplus R_n(V_{2,0},\chi).$$

If $m = 2n$ and $\chi|_{F^\times} = 1$, let V_1 be the split form of dimension $2n$. Then

$$M^*(s_0,\chi)(I_n(s_0,\chi)) = M^*(s_0,\chi)(R_n(V_1,\chi)) = R_n(0,\chi).$$

Proof: First suppose that $n \leq m \leq 2n$ and let V be any Hermitian space of dimension m which has a complement V_0. We are fixing χ with $\chi|_{F^\times} = \epsilon_{E/F}^m$, and so we omit mention of χ for brevity. Since $R_n(V)$ is the image of $S(V^n)$ under the mapping $\varphi \mapsto \omega(g)\varphi(0)$, by Lemma 5A.1 of the appendix to this section, we have

$$\dim \operatorname*{Hom}_G(R_n(V), I_n(-s_0)) \leq 1. \tag{5.12}$$

But by Proposition 5.7,

$$1 \leq \dim \operatorname*{Hom}_G(R_n(V), R_n(V_0)) \leq \dim \operatorname*{Hom}_G(R_n(V), I_n(-s_0)) \leq 1, \tag{5.13}$$

so that both dimensions are 1. Now suppose we can show that $M_n^*(s_0)R_n(V) \neq 0$. Then $M_n^*(s_0)$ must be the unique non-zero operator in $\operatorname{Hom}_G(R_n(V), I_n(-s_0))$ and hence also in $\operatorname{Hom}_G(R_n(V), R_n(V_0))$. By the irreducibility of $R_n(V_0)$, we could then conclude that $M_n^*(s_0)R_n(V) = R_n(V_0)$, as desired.

Hence it suffices to prove that $M_n^*(s_0)R_n(V) \neq 0$ for each space V which has a complement. For the moment, suppose that $n \leq m < 2n$, so that each space V_i has a complement ($i = 1, 2$). By Proposition 5.6, since $I_n(s_0)$ is generated by the two spaces $R_n(V_i)$, and since $M_n^*(s_0) \neq 0$, we know that $M_n^*(s_0)$ is non-zero on one of the two subspaces. Without loss, we then suppose that $M_n^*(s_0)R_n(V_1) = R_n(V_{1,0})$, and prove the corresponding identity for V_2.

First, suppose that $\chi|_{F^\times} = \epsilon_{E/F}$, so that m is odd. In this case, fix an element $a \in F^\times \sim N(E^\times)$. Choose a basis $\{v_1, \ldots, v_m\}$ for V_1, and note that the matrices $Q_1 = ((v_i, v_j)_1)$ and $Q_2 = (a(v_i, v_j)_1)$ have determinants which differ by $a \cdot (a^{\frac{m-1}{2}})^2$. Hence when considered in $F^\times/N(E^\times)$, $\det(Q_1) \neq \det(Q_2)$. In other words, we may take V_2 to have the same underlying space as V_1, but with Hermitian form equal to $a \cdot (\, , \,)_1$. As in [19], one then checks that conjugation by

$$\tau_a = \begin{pmatrix} 1_n & 0 \\ 0 & a \cdot 1_n \end{pmatrix} \in \mathrm{GL}(2n, E) \tag{5.14}$$

induces an outer automorphism $\mathrm{Ad}(\tau_a)$ of G_n which preserves the space $I_n(s_0)$ and intertwines the subspace $R_n(V_1)$ with $R_n(V_2)$, and similarly for $R_n(V_{1,0})$ and $R_n(V_{2,0})$. We may also compute that for $\Phi \in I_n(s_0)$,

$$M_n^*(s_0)(\mathrm{Ad}(\tau_a))\Phi = \chi(a^n)|a|^{ns_0}(\mathrm{Ad}(\tau_a))(M_n^*(s_0)\Phi). \tag{5.15}$$

Hence if $\Phi \in R_n(V_1)$ and $M_n^*(s_0)\Phi \neq 0$, then $\Phi_a = \mathrm{Ad}(\tau_a)\Phi \in R_n(V_2)$ and $M_n^*(s_0)\Phi_a \neq 0$ also. It follows that $M_n^*(s_0)R_n(V_2) = R_n(V_{2,0})$.

On the other hand, if $\chi|_{F^\times} = 1$, then for any s,

$$I_n(s, \chi) \simeq \mathrm{Ind}_{P_0}^K(\chi) \simeq \chi_G \otimes \mathrm{Ind}_{P_0}^K(1) \tag{5.16}$$

as representations of $K = G \cap \mathrm{GL}(2n, \mathcal{O}_E)$, writing $P_0 = P \cap K$ temporarily. We now claim that $I_n(s)$ is multiplicity-free as a K-module. By the above, this is true if and only if the algebra $\mathrm{Hom}_K(A, A)$ is commutative, where $A = \mathrm{Ind}_{P_0}^K(1)$. Now, there is an algebra isomorphism from the Hecke algebra $\mathcal{H}(K//P_0)$ to $\mathrm{Hom}_K(A, A)$ given by $f \mapsto f*$, where $f*$ stands for convolution with $f \in \mathcal{H}(K//P_0)$. In order to prove that the Hecke algebra with convolution is commutative, it suffices (as in Rallis [26]) to find an anti-involution τ of K satisfying $\tau(k) = p_1 k p_2$ for some $p_1, p_2 \in P_0$ depending on $k \in K$. This is provided by modifying the proof in the appendix to §4 of [26], using the definition

$$\tau(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\bar{k} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.17}$$

Hence $I_n(s,\chi)$ is multiplicity-free as a K-module if $\chi|_{F^\times} = 1$. Next, let Θ be a K-type in $I_n(s)$. Then, when restricted to Θ, $M_n^*(s)|_\Theta = C_\Theta(s)\cdot \mathrm{Id}_\Theta$, for some entire function $C_\Theta(s)$. By the proof of Proposition 3.2, it is immediate that

$$C_\Theta(s)\cdot C_\Theta(-s) = \frac{\gamma(s,\chi)}{a(s,\chi)}\frac{\gamma(-s,\chi)}{a(-s,\chi)} \quad \text{for all } \Theta.$$

But now suppose that $n < m < 2n$, so that $0 < m' = 2n - m < n$. Then $s_0 > 0$ and $\frac{\gamma(s,\chi)}{a(s,\chi)} = 0$ at $s = s_0$, while it is non-zero at $s = -s_0$ (by Lemma 3.5). So $C_\Theta(s)\cdot C_\Theta(-s)$ has a simple zero at $s = s_0$.

The proof of the following result follows closely the proofs of Theorem 2.8 of [14] and Proposition 4.4 of [17].

Lemma 5.9: *Suppose that $\chi|_{F^\times} = \epsilon_{E/F}^m$ and that V is a non-degenerate Hermitian vector space with $\dim_E(V) = m$. Let $s_0 = \frac{m-n}{2}$. Then set*

$$d = \dim_{\mathbb{C}} \operatorname*{Hom}_{U(V)\times U(n,n)}(\mathcal{S}(V^n), 1\otimes I_n(-s_0,\chi)),$$

where the Weil representation acts on $\mathcal{S}(V^n)$ as described in Lemma 4.1. Let l be the Witt index of V. The following inequalities hold:

(1) *if $m < n$, then $d = 0$.*
(2) *If $n \le m < 2n$, then $d \le 1$.*
(3) *If $2n \le m$, then* $\begin{cases} d = 0 & \text{if } m = 2n \text{ and } V \text{ is non-split,} \\ d \le 1 & \text{otherwise.}\end{cases}$

Note: we actually prove that $d = 1$ in all cases above in which this is allowed.

By Lemma 5.9,

$$\dim \operatorname*{Hom}_G(R_n(V_{i,0}), I_n(s_0)) = 0 \qquad (i = 1,2), \tag{5.18}$$

which implies that, for any $\Theta \in R_n(V_{i,0})$, $C_\Theta(-s_0) = 0$, so that $C_\Theta(s_0) \neq 0$. Hence $M_n^*(s_0)$ hits everything in $R_n(V_{1,0}) \oplus R_n(V_{2,0})$. Since we assumed that $M_n^*(s_0)R_n(V_1) = R_n(V_{1,0})$, this implies that in fact $M_n^*(s_0)R_n(V_2)$ must be non-zero, as desired.

Next, suppose that $m = n$ and m is even. Then $I_n(0) = R_n(V_1) \oplus R_n(V_2)$. As above, we see that $C_\Theta(0)^2 \neq 0$ for any K-type Θ in $I_n(0)$, so that $M_n^*(0)$ is an isomorphism. By our assumption, clearly $M_n^*(0)R_n(V_2) = R_n(V_2)$.

Finally, suppose that $m = 2n$ and V is split. Then V has as its complementary space $V_0 = 0$, and we recall that $I_n(s_0) = R_n(V)$ in this case. But then as at the beginning of our proof,

$$\dim \operatorname*{Hom}_G(I_n(s_0), R_n(0)) = \dim \operatorname*{Hom}_G(I_n(s_0), I_n(-s_0)) = 1, \tag{5.19}$$

and so both spaces are spanned by $M_n^*(s_0)$. Hence $M_n^*(s_0)R_n(V) = R_n(V_0)$. ∎

PROPOSITION 5.10: *Suppose that $n \le m < 2n$ and that $\chi|_{F^\times} = \epsilon_{E/F}^m$. Then*

$$\ker M^*(s_0, \chi) = R_n(V_1, \chi) \cap R_n(V_2, \chi).$$

Note that this is 0 when $s_0 = 0$.

Proof: We are assuming that $0 \le s_0 < \frac{n}{2}$. First, recall that if $s_0 = 0$, then $I_n(0,\chi) = R_n(V_1,\chi) \oplus R_n(V_2,\chi)$ is the sum of two irreducible, inequivalent subspaces. We noted in the preceding proof that $M_n^*(0,\chi)$ is an isomorphism, so the kernel is 0 and we are finished in this case.

The proof is by induction on n, so we note that if $n = 1$ and $n \le m < 2n$, then $s_0 = 0$, and we are through. So we assume the result for G_{n-1}, and for all m with $n-1 \le m < 2n-2$. If $n < m < 2n$, then by Proposition 5.6, $I_n(s_0) = R_n(V_1) + R_n(V_2)$, suppressing mention of χ. Here $V_1 \not\cong V_2$, and both spaces have complements. Also by the preceding Proposition, we have an exact sequence

$$0 \to Y_n(s_0) \to I_n(s_0) \xrightarrow{\lambda} R_n(V_{1,0}) \oplus R_n(V_{2,0}) \to 0, \tag{5.20}$$

where λ is induced by $M_n^*(s_0)$, and $Y_n(s_0)$ is its kernel. Applying the (exact) N_1-Jacquet functor to this, we obtain

$$0 \to Y_n(s_0)_{N_1} \to I_n(s_0)_{N_1} \xrightarrow{\lambda_{N_1}} R_n(V_{1,0})_{N_1} \oplus R_n(V_{2,0})_{N_1} \to 0. \tag{5.21}$$

The exact sequences of Proposition 5.3 (1) and (2) split, so by considering the characters of $\mathrm{GL}(1) \subset M_1$, it is clear that λ_{N_1} segregates into $\lambda_1 \oplus \lambda_2$, where

$$\begin{aligned} &\lambda_1\colon I_{n-1}(s_0 - \tfrac12) \to R_{n-1}(V'_{1,0}) \oplus R_{n-1}(V'_{2,0}) \to 0, \\ &\lambda_2\colon I_{n-1}(s_0 + \tfrac12) \to R_{n-1}(V_{1,0}) \oplus R_{n-1}(V_{2,0}) \to 0. \end{aligned} \tag{5.22}$$

Now $\lambda_1 \in \mathrm{Hom}_{G_{n-1}}(I_{n-1}(s_0 - \frac12), I_{n-1}(\frac12 - s_0))$, and by considering Theorem 2.2 (1) and Proposition 5.8, one can see that the basis for this Hom space can be constructed from $M^*(s_0 - \frac12)$ composed with projection onto the two subrepresentations $R_{n-1}(V'_{i,0})$. Since λ_1 is surjective, it must in fact have the same kernel as $M^*(s_0 - \frac12)$. The same statement is true for λ_2 and $M^*(s_0 + \frac12)$. But now $n < m < 2n$, so that $n-1 \leq m-2 < 2(n-1)$, and so by our induction hypothesis, we have

$$\ker(\lambda_1) = R_{n-1}(V_1') \cap R_{n-1}(V_2')$$

and

$$\ker(\lambda_2) = R_{n-1}(V_1) \cap R_{n-1}(V_2),$$

so that

$$Y_n(s_0)_{N_1} \simeq (R_{n-1}(V_1') \cap R_{n-1}(V_2')) \oplus (R_{n-1}(V_1) \cap R_{n-1}(V_2)).$$

As in [19], this implies that $Y_n(s_0) = R_n(V_1) \cap R_n(V_2)$. ∎

6. Exponents and the intertwining operators

In order to complete our information about the composition series of $I(s,\chi)$ and to compute the action of the normalized intertwining operator $M^*(s,\chi)$ on the constituents, we consider the exponents of the constituents of $I_n(s,\chi)$. Let B be the standard Borel subgroup of G containing the maximal torus

$$A = \{m(a) \in G \mid a = \mathrm{diag}(a_1, \ldots, a_n) \text{ for } a_i \in E^\times\}, \tag{6.1}$$

and define the character ρ_B of A by $m(a)^{\rho_B} = \delta_B(a)^{\frac{1}{2}}$. If $m(a) \in A$ is as above, one computes that

$$m(a)^{\rho_B} = |a_1|_E^{n-\frac{1}{2}} |a_2|_E^{n-\frac{3}{2}} \cdots |a_n|_E^{\frac{1}{2}}, \tag{6.2}$$

which we represent by writing

$$\rho_B = \rho_B(n) = (n - \tfrac{1}{2}, n - \tfrac{3}{2}, \ldots, \tfrac{1}{2}).$$

Let $U = U_n$ be the unipotent radical of B. We will compute the exponents of the constituents π of $I_n(s,\chi)$ along B, that is, the characters μ of A such that $m(a)^{\mu+\rho_B}$ occurs in a decomposition of the Jacquet module π_U into a sum of generalized eigenspaces. As in [19], we compute π_U by noting that $\pi_U \simeq (\pi_{N_1})_{U\cap M_1}$, and applying the $U \cap M_1$ Jacquet functor to the exact sequences of Proposition 5.3. Note that $U_n \cap M_1 \simeq U_{n-1} \subset G_{n-1}$ under the identification $M_1 \simeq \mathrm{GL}(1,E) \times G_{n-1}$, and so as $\mathrm{GL}(1) \times A_{n-1} = A_n$ modules, we have

$$\begin{aligned}(6.3)\quad 0 \longrightarrow \check{\chi}|\,|^{-s+\frac{n}{2}} \otimes I_{n-1}(s-\tfrac{1}{2},\chi)_{U_{n-1}} &\longrightarrow I_n(s,\chi)_{U_n} \\ &\longrightarrow \chi|\,|^{s+\frac{n}{2}} \otimes I_{n-1}(s+\tfrac{1}{2},\chi)_{U_{n-1}} \longrightarrow 0.\end{aligned}$$

There is a one-to-one correspondence between the exponents μ of $I_n(s,\chi)$ and sequences $(c_1,\ldots,c_n)$ of length n which occur as you move successively either left or right in the exact sequence above, setting c_k equal to the GL(1) exponent minus $\rho_B(k)$ on the k^{th} move. But in fact, one obtains $1-s-(\frac{n+1}{2})$ on the first left move, regardless of when this occurs, and similarly $k-s-(\frac{n+1}{2})$ on the k^{th} left move. The k^{th} right move always produces a component $s+(\frac{n+1}{2})-n+(k-1)$. So the exponents are all "shuffles" of the components resulting from left moves with those resulting from right moves. Here we define a shuffle as follows: If an ordered sequence E of length n is divided into two ordered subsequences, $E=(A;B)$, then a shuffle of E is a sequence which results from a permutation of E which preserves the relative ordering of the elements of A, and also those of B.

PROPOSITION 6.1: *The representation $I_n(s,\chi)$ has 2^n exponents, which may be described as follows. For each r, with $0\le r\le n$, every shuffle of the n-tuple*

$$\left(1-s-\left(\frac{n+1}{2}\right),2-s-\left(\frac{n+1}{2}\right),\ldots,r-s-\left(\frac{n+1}{2}\right);\right.$$
$$\left.s+\left(\frac{n+1}{2}\right)-n,\ldots,s+\left(\frac{n+1}{2}\right)-r-1\right)$$

is an exponent. Moreover, these exponents are counted with multiplicity.

When we compute the exponents for the spaces $R_n(V,\chi)$, the only difference is that the maximum number of possible left moves is limited by the number of hyperplanes which can be removed from V.

PROPOSITION 6.2: *Let l be the Witt index of the Hermitian space V (the dimension of a maximal isotropic subspace of V). Then the exponents of $R_n(V,\chi)$ are obtained as follows. For each r with $0\le r\le \min(n,l)$, every shuffle of the n-tuple*

$$\left(1-\left(\frac{m+1}{2}\right),2-\left(\frac{m+1}{2}\right),\ldots,r-\left(\frac{m+1}{2}\right);\right.$$
$$\left.\left(\frac{m+1}{2}\right)-n,\ldots,\left(\frac{m+1}{2}\right)-r-1\right)$$

is an exponent of $R_n(V,\chi)$; and, when counted with multiplicity, these are all of the exponents.

Now we must study the combinatorics of the exponents to proceed further. Let $\rho = \frac{m+1}{2}$ for ease of notation, and for $0 \leq r \leq n$, let

$$A_r = (1-\rho, 2-\rho, \ldots, r-\rho), \quad B_r = (\rho-n, \rho-n+1, \ldots, \rho-(r+1)). \tag{6.4}$$

As in Propositions 6.1 and 6.2, we see that the shuffles of $E_r = (A_r; B_r)$ for $0 \leq r \leq n$ correspond to the exponents of the $I_n(s,\chi)$ at $s_0 = \frac{m-n}{2}$.

Now consider the exponents corresponding to the submodules $R_n(V_i,\chi)$ for $n \leq m \leq 2n$. With m and n in this range, suppose that m is even. Then we let V_1 be the split space composed of $\frac{m}{2}$ hyperbolic planes, so that the Witt index of V_1 is $\frac{m}{2}$. The other space V_2 can be taken to be the direct sum of the unique 2-dimensional anisotropic space with $\frac{m}{2}-1$ hyperbolic planes, so that the Witt index is $\frac{m}{2}-1$. If m is odd, then the two possibilities both have Witt index $\frac{m-1}{2}$, and result from adding this many hyperbolic planes to an anisotropic space of dimension 1, whose form is specified by $1 \in F^\times$ for V_1, or $\alpha \in F^\times \sim N(E^\times)$ for V_2. So, to be uniform, we define

$$I = \{\text{all shuffles of } E_r \text{ for } 0 \leq r \leq n, \text{ counted with multiplicity}\}, \tag{6.5}$$

and (omitting mention of multiplicities from now on)

$$R_1 = \left\{\text{all shuffles of } E_r \text{ for } 0 \leq r \leq \left[\frac{m}{2}\right]\right\}, \quad R_2 = \left\{\text{all shuffles of } E_r \text{ for } 0 \leq r \leq \left[\frac{m-1}{2}\right]\right\}, \tag{6.6}$$

so that I represents all exponents of $I_n(s_0,\chi)$ (and in fact, those of $I_n(-s_0,\chi)$ as well), and R_i represents the exponents of $R_n(V_i,\chi)$ for $i=1,2$ by the remarks above, and Proposition 6.2. We then let

$$S_1 = \left\{\text{all shuffles of } E_r \text{ for } \left[\frac{m+2}{2}\right] \leq r \leq n\right\}, \quad S_2 = \left\{\text{all shuffles of } E_r \text{ for } \left[\frac{m+1}{2}\right] \leq r \leq n\right\}, \tag{6.7}$$

so that clearly $I = R_1 + S_1 = R_2 + S_2$, where $+$ denotes the disjoint union, increasing multiplicities appropriately (and below, $-$ denotes the difference of sets, again taking multiplicities into account). One then checks that S_1 represents the exponents of $R_n(V_{2,0},\chi)$, and similarly for S_2 and $R_n(V_{1,0},\chi)$. In particular, note that when $m=2n$, V_2 has no complementary space, and correspondingly, $S_1=\emptyset$. We also let D denote the set of exponents of $R_n(V_1,\chi)\cap R_n(V_2,\chi)$. We then have the following combinatorial facts:

PROPOSITION 6.3: *Let $n\le m\le 2n$. Then*

(1) *$S_2\subset R_1$ and $S_1\subset R_2$.*
(2) *$I-S_1-S_2=R_1-S_2=R_2-S_1$, and if we call this last set D, then all sequences in D occur with multiplicity one.*
(3) *$D\cap S_i=\emptyset$ for $i=1,2$.*

Proof: For $n\le m<2n$, $M_n^*(s_0,\chi)$ induces an isomorphism

$$R_n(V_1,\chi)/(R_n(V_1,\chi)\cap R_n(V_2,\chi))\simeq R_n(V_{1,0},\chi) \tag{6.8}$$

(by Propositions 5.8 and 5.10), and hence of the corresponding U-Jacquet modules. This implies that $R_1-D=S_2$, which (with the analogous result $R_2-D=S_1$) is equivalent to all statements from (1) and (2) save the one about multiplicity. (1) and (2) are also trivially checked in the case $m=2n$. The assertion of multiplicity one for D together with (3) can be handled purely combinatorially (as can the other results of (1) and (2) for that matter) by following the blueprint of Proposition 6.2 of [19]. ∎

PROPOSITION 6.4: *If $n<m\le 2n$ and $\chi|_{F^\times}=\epsilon_{E/F}^m$, then*

$$\ker M_n^*(-s_0,\chi)=R_n(V_{1,0},\chi)\oplus R_n(V_{2,0},\chi),$$
$$\mathrm{image} M_n^*(-s_0,\chi)=R_n(V_1,\chi)\cap R_n(V_2,\chi).$$

In particular, if $m=2n$, then

$$\ker M_n^*(-\frac{n}{2},\chi)=R_n(0,\chi),$$
$$\mathrm{image}\ M_n^*(-\frac{n}{2},\chi)=R_n(V_2,\chi).$$

Proof: First, we may write $I(s_0)_U=X\oplus Y$, where Y is the direct sum of all generalized eigenspaces having exponents in D, and X is the sum of all spaces having exponents in $I-D$. Since all exponents of Y have multiplicity one, and are distinct from those of X, there is a small neighborhood of s_0 on which we may write $I(s)_U=X(s)\oplus Y(s)$, where the exponents of $X(s)$ and $Y(s)$ interpolate those of X and Y, and they still have the characteristics just mentioned. Since I also represents the exponents of $I_n(-s_0)$, we may do the same for $I(s)_U$ in a neighborhood of $-s_0$. Now fix any character $\mu \in D$, and extend the corresponding eigenvector in $Y(s_0)$ to an eigenvector $f_1(s) \in Y(s)$ for $\mu(s)$ by choosing a representative in $I_n(s_0, \chi)$ and using this to define a standard section in $I_n(s, \chi)$. $f_1(s)$ may then be taken to be the image under the U-Jacquet functor of this section. Do the same for the eigenvector in $Y(-s)$ corresponding to $\mu(-s)$, so that $f_2(-s) \in Y(-s)$. Now, we have induced operators $M_n^*(\pm s, \chi)_U$ which must preserve the spaces $Y(\pm s) \subset I_n(\pm s)_U$, respectively. So

$$
\begin{aligned}
M_n^*(s)_U f_1(s) &= c(s) f_2(-s) \\
M_n^*(-s)_U f_2(-s) &= c'(s) f_1(s)
\end{aligned}
\tag{6.9}
$$

for some holomorphic functions $c(s)$ and $c'(s)$ defined in a neighborhood of s_0. Since $f_1(s_0)$ transforms according to $\mu \in D$ under the action of A, we see that $M_n^*(s_0)_U f_1(s_0) = 0$ by recalling that the image of $M_n^*(s_0)$ has exponents equal to $S_1 \cup S_2$, which is disjoint from D (see Proposition 5.8). Hence $c(s_0) = 0$. But we also see that

$$
c(s)c'(s) = \frac{\gamma(s)}{a(s)} \cdot \frac{\gamma(-s)}{a(-s)},
\tag{6.10}
$$

by the usual trick, and, as in the proof of Proposition 5.8, the expression above has a simple zero at $s = s_0$. Hence $c'(s_0) \neq 0$, so that $M_n^*(-s_0)_U$ is non-zero on $Y(-s_0)$. But by Lemma 5A.1, $R_n(V_{1,0}) \oplus R_n(V_{2,0}) \subset \ker M_n^*(-s_0)$, so that in fact these last spaces are equal. Finally, it follows from the disjointness of D and the S_i that the image of $M_n^*(-s_0)_U$ is $Y(s_0) = (R_n(V_1) \cap R_n(V_2))_U$, from which we conclude that image $M^*(-s_0, \chi) = R_n(V_1, \chi) \cap R_n(V_2, \chi)$. ∎

PROPOSITION 6.5: *If $n < m \leq 2n$ and $\chi|_{F^\times} = \epsilon_{E/F}^m$, then $R_n(V_1, \chi) \cap R_n(V_2, \chi)$ is the unique irreducible submodule of $I_n(s_0, \chi)$.*

Proof: Suppose that $W \subset R_n(V_1, \chi) \cap R_n(V_2, \chi) \subset I_n(s_0, \chi)$ is irreducible and non-zero. Then as in the proof of Theorem 3.3, we may construct an operator $\varphi_1\colon I_n(-s_0, \chi^{-1}) \to W^\delta \to 0$. But consider

$$
\varphi_2\colon I_n(-s_0, \check{\chi})^\delta \xrightarrow{\sim} I_n(-s_0, \chi^{-1}) \longrightarrow W^\delta \longrightarrow 0.
\tag{6.11}
$$

In fact, $\varphi_2\colon I_n(-s_0, \check{\chi}) \to W$ is a surjective G-morphism, and we may regard φ_2 as lying in $\mathrm{Hom}_G(I_n(-s_0, \check{\chi}), I_n(s_0, \chi))$. But now $-\frac{n}{2} < -s_0 < 0$ so that this Hom space is one-dimensional by Proposition 2.2, and hence φ_2 is a non-zero multiple of $M_n^*(-s_0, \check{\chi})$. By the preceding proposition, this has image $R_n(V_1,\chi) \cap R_n(V_2,\chi) = W$, and so this last space is irreducible. Uniqueness follows by Proposition 2.3. ∎

PROPOSITION 6.6: *If* $\chi|_{F^\times} = 1$, *then* $\ker M^*(\frac{n}{2},\chi) = R_n(V_2,\chi)$.

Proof: This is easily seen from the fact that all exponents of $I_n(\frac{n}{2},\chi)$ have multiplicity 1, since $I = S_2 \cup D$ with $D = S_1 =$ the set of exponents of $R_n(V_2,\chi)$, $|S_2| = 1$, and $S_2 \cap D = \emptyset$. ∎

7. Sketch of proof for GL$(2n, F)$

Finally, we sketch the proof of Theorem 1.3 of the introduction, which concerns the induced representation $I(s,\sigma) = \mathrm{Ind}_P^G(\sigma_1|\,|_F^s \otimes \sigma_2|\,|_F^{-s})$ of $G = \mathrm{GL}(2n,F)$, where P is the standard $\mathrm{GL}(n)\times\mathrm{GL}(n)$ maximal parabolic. Let all notation be as in the paragraph immediately preceding Theorem 1.3. We also use the following convention from [2] and [36]: given a character η of $F^\times$ and $s \in \mathbb{C}$, let $\eta(s)$ be the quasi-character $x \mapsto \eta(x)|x|_F^s$. Consider the quasi-characters of $\mathrm{GL}(1,F)\times\cdots\times\mathrm{GL}(1,F)$ (n copies) given by

$$\triangle_1 = (\sigma_1(s+\tfrac{1-n}{2}), \sigma_1(s+\tfrac{3-n}{2}), \dots, \sigma_1(s+\tfrac{n-1}{2})), \qquad \text{and}$$
$$\triangle_2 = (\sigma_2(-s+\tfrac{1-n}{2}), \sigma_2(-s+\tfrac{3-n}{2}), \dots, \sigma_2(-s+\tfrac{n-1}{2}))$$

(we think of these either as sets of quasi-characters of $F^\times$, or as quasi-characters of the product of n copies of $\mathrm{GL}(1,F)$). In the notation of Bernstein and Zelevinsky, these are *segments*: sets of quasi-characters (or more generally, supercuspidals) which are of the form $(\rho, \rho(1), \rho(2), \dots, \rho(k))$ for $k \geq 0$, where ρ is a quasi-character of $F^\times$ (respectively, a supercuspidal). Corollary 2.9 of [36] tells us that for any segment $\triangle$ composed of n quasi-characters, the representation $\mathrm{Ind}_B^{\mathrm{GL}(n,F)}(\triangle)$ has a unique irreducible subrepresentation $Z(\triangle)$. Here B is the standard (upper triangular) Borel subgroup of $\mathrm{GL}(n,F)$. It is easy to see (example 3.2 of [36]) that in our situation

$$Z(\triangle_1) = \sigma_1|\,|_F^s \in \mathrm{Irr}(\mathrm{GL}(n,F)), \qquad \text{and}$$
$$Z(\triangle_2) = \sigma_2|\,|_F^{-s} \in \mathrm{Irr}(\mathrm{GL}(n,F)),$$

so that our representation $I(s,\sigma)$ is in fact equal to $\mathrm{Ind}_P^G(Z(\triangle_1)\otimes Z(\triangle_2))$. By Theorem 4.2 of [36], this representation is reducible if and only if the segments

$\triangle_1$ and $\triangle_2$ are *linked*: that is, if and only if $\triangle_1 \not\subset \triangle_2$, and $\triangle_1 \not\subset \triangle_2$, and $\triangle_1 \cup \triangle_2$ is a segment. Given our normalization of s and σ, it is easily checked that this happens precisely for (s,σ) as given in item (1) of our Theorem 1.3. In addition, the length of the composition series and the identification of one constituent as induced from a certain maximal parabolic are given in Proposition 4.6 of [36].

Now we assume, for the moment, that $\sigma = \sigma_1 = \sigma_2$, and describe the relationship between the constituents at the points of reducibility and the Weil representation. We begin by considering the ring $E = F \oplus F$ with componentwise addition and multiplication, and with an involution given by $x = [x_1, x_2] \mapsto \bar{x} = [x_2, x_1]$. We may identify F with the fixed field of the involution, and so naturally define a trace operator from E to F via $T([x_1,x_2]) = x_1 + x_2$. Define two E-modules V and W with 'Hermitian' and 'skew-Hermitian' forms (respectively) via

$$V = E^m \quad \text{(col. vectors)} \qquad \text{and} \qquad (a,b) = [\,{}^t a_1 \cdot b_2,\ {}^t a_2 \cdot b_1],$$

where $a, b \in V$ are written in the form $a = [a_1, a_2] \in F^m \oplus F^m$,

$$W = E^{2n} \quad \text{(row vectors)} \qquad \text{and} \qquad \langle c, d\rangle = [c_1 J\, {}^t d_2, c_2 J\, {}^t d_1],$$

for $c, d \in W$ of the form $c = [c_1, c_2] \in F^{2n} \oplus F^{2n}$. Here J is as in equation (1.1). It is easily seen that the isometry groups H and G of the respective spaces V and W satisfy $H \simeq \mathrm{GL}(m,F)$ and $G \simeq \mathrm{GL}(2n,F)$ by 'projection on the first factor'. As in the unitary case, we set $\mathbf{W} = V \otimes_E W$ with a symplectic form over F given by $\ll , \gg = T((\,,\,) \otimes \overline{\langle\,,\,\rangle})$. Then $H \times G$ forms a dual pair in $\mathrm{Sp}(\mathbf{W})$, and, corresponding to the complete polarization $W = E^n \oplus E^n$, there is a natural complete polarization $\mathbf{W} = V^n \oplus V^n$ and a Schrodinger model $\mathcal{S}(V^n)$ for the Weil representation of the metaplectic cover of $\mathrm{Sp}(\mathbf{W})$. This projective representation now splits over both G and H: we write ω for the restriction of the Weil representation to $H \times G$ acting in the space $\mathcal{S}(V^n)$. One then checks that the mapping

$$\begin{aligned} \mathcal{S}(V^n) &\longrightarrow \mathrm{Ind}_P^G(|\,|_F^{\frac{m-n}{2}} \otimes |\,|_F^{\frac{n-m}{2}}) \quad \text{given by} \\ \varphi &\longmapsto \{g \mapsto \omega(g)\varphi(0)\}, \end{aligned}$$

is well-defined, G-intertwining, and intertwines the natural action of H on the left with the trivial action of H on the right. Setting $s_0 = \frac{m-n}{2}$, we may then twist the image of the above map by σ so that it lies in $I(s_0, \sigma)$. We denote this twisted image in $I(s_0,\sigma)$ by $R_n(m,\sigma)$. Now for $1 \leq m$, there are only two possibilities for $R_n(m,\sigma)$: all of $I(s,\sigma)$, or the unique irreducible submodule guaranteed by Bernstein and Zelevinsky.

To check that $R_n(m,\sigma) = I(s_0,\sigma)$ when $m \geq n$, we use the argument of Proposition 4.3 (3) above. Because of the similar formulas for the Weil representation, this amounts to checking that the moment map

$$\mu\colon V^n_{\mathrm{sub}} \longrightarrow X$$

is surjective for $m \geq n$. Here μ is defined on all of V^n just as in equation (4.8), and V^n_{sub} is the subset of V^n on which μ is submersive: which is to say $V^n_{\mathrm{sub}} = \{x \in V^n \mid d\mu_x \text{ is surjective}\}$. But there are natural isomorphisms $V^n \simeq M(m,n,F) \times M(m,n,F)$ and $X \simeq M(n,F)$ which allow us to model μ using

$$\begin{aligned} \mu'\colon M(m,n,F)^2 &\longrightarrow M(n,F), \\ [a,b] &\longmapsto {}^t a \cdot b. \end{aligned}$$

For $m \geq n$, the set of $[a,b]$ for which $d\mu'_{[a,b]}$ is submersive consists of those for which the matrix $(a\colon b) \in M(m,2n,F)$ has rank at least n. It is then easy to see that $\mu\colon V^n_{\mathrm{sub}} \to X$ is surjective, and hence that $R_n(m,\sigma) = I(s_0,\sigma)$. Also, by using the method of Proposition 3.1 of [19], one may show that $R_n(m,\sigma)$ is irreducible and unitarizable if $1 \leq m \leq n$ (even as a representation of P), so that it must in fact be the unique irreducible subrepresentation of $I(s_0,\sigma)$ in these cases. The case $m = 0$ is obvious.

Next, we briefly describe the normalization of the intertwining operator $M(s,\sigma)\colon I(s,\sigma) \to I(-s,\check{\sigma})$, which is defined in equation (1.9) of Theorem 1.3. We again let $\sigma = (\sigma_1,\sigma_2)$ be arbitrary. As in the unitary case, it is useful to study the operator $M(s,\sigma)$ applied to special sections of the form $\Phi_\varphi(s)$, for $\varphi \in \mathcal{S}(X)$ (see equation (3.12)), writing $X = M(n,F)$. An easy computation along the lines of Proposition 3.1 shows that

$$M(s,\sigma)\Phi_\varphi(w_n) = \int_X \sigma_1(x)\sigma_2(x)^{-1}|x|_F^{2s}\,\varphi(x)\,\frac{dx}{|x|_F^n},$$

and so we are led to study zeta integrals of the form

$$Z(s,\chi,\varphi) = \int_X \chi(x)|x|_F^{s}\,\varphi(x)\,\frac{dx}{|x|_F^n},$$

for characters χ of $F^\times$. Here, X is a vector space on which $\mathrm{GL}(n,F)$ acts by left multiplication, and the non-singular matrices $Y \subset X$ form a single $\mathrm{GL}(n,F)$-orbit. So we again have a pre-homogeneous vector space $(\mathrm{GL}(n,F),X,Y)$ and a functional equation for $Z(s,\chi)$ as in [8]. Igusa computed the gamma factor in a second paper [9], and, translating his results, we have the following functional equation:

$$Z(s,\chi,\varphi) = \Gamma(s,\chi)\cdot Z(n-s,\chi^{-1},\hat{\varphi}),$$

where $\hat{\varphi}$ is defined appropriately (using our additive character ψ), and where the gamma factor is given by

$$\Gamma(s,\chi) = \prod_{i=0}^{n-1} \rho_F(s-i,\chi).$$

By Rallis's Lemma, the operator $M(s,\sigma)$ has the same zero and pole behavior as $Z(2s,\chi_\sigma)$, writing $\chi_\sigma = \sigma_1/\sigma_2$, and we may analyze this last by the method of [24]. From this, we deduce that the correct normalization consists of dividing by the *full* numerator of $\Gamma(2s,\chi_\sigma)$ (even though all but one of the zeta functions appearing there, cf. equation (3.11), could be canceled with those in the denominator, leaving exponential factors). So the correct normalization of $M(s,\sigma)$ is as given in Theorem 1.3. Also note that we may define a generalized Whittaker integral on $I(s,\sigma)$ via

$$(W(s)\Phi)(g) = \int_X \Phi(w_n n(b)g,s)\psi(\mathrm{tr}(b))\,db$$

(there is just one to consider). This has a functional equation

$$W(-s)\circ M(s,\sigma) = \Gamma(2s,\chi_\sigma)\cdot W(s),$$

with the same gamma factor given above. As before, we see that

$$M^*(-s,\check{\sigma})\circ M^*(s,\sigma) = \frac{\Gamma(2s,\chi_\sigma)}{a(s,\chi_\sigma)}\cdot\frac{\Gamma(-2s,\chi_\sigma^{-1})}{a(-s,\chi_\sigma^{-1})}, \tag{7.1}$$

and so $M^*(s,\sigma)$ cannot be surjective if (s,σ) is a point of reducibility of $I(s,\sigma)$ (these points are precisely the zeros of the right hand side of (7.1)). This tells us that the kernel and image of $M^*(s,\chi)$ must be as claimed, since $M^*(s,\sigma)$ is non-zero and non-surjective at the points of reducibility.

References

[1] J. Bernstein and A. V. Zelevinsky, *Representations of the group* GL(n, F), *where* F *is a nonarchimedean local field,* Russian Mathematical Surveys **31** (1976), 1–68.

[2] J. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups, I,* Annales Scientifiques de l'École Normale Supérieure **4, t. 10** (1977), 441–472.

[3] S. Gelbart, J. Rogawski and D. Soudry, *Endoscopy and theta-liftings for the unitary group in three variables,* preprint, 1994.

[4] S. Gelbart and F. Shahidi, *Analytic Properties of Automorphic L-functions,* Perspectives in Math., Vol. 6, Academic Press, Boston, 1988.

[5] R. Gustafson, *The degenerate principal series of* $Sp(2n)$, Memoirs of the American Mathematical Society **248** (1981), 1–81.

[6] M. Harris, *L-functions of* 2×2 *unitary groups and factorization of periods of Hilbert modular forms,* Journal of the American Mathematical Society **6** (1993), 637–719.

[7] M. Harris, S. Kudla and W. J. Sweet, Jr., *Theta dichotomy for unitary groups,* to appear in the Journal of the American Mathematical Society.

[8] J. Igusa, *Some results on p-adic complex powers,* American Journal of Mathematics **106** (1984), 1013–1032.

[9] J. Igusa, *On functional equations of complex powers,* Inventiones mathematicae **85** (1986), 1–29.

[10] N. Jacobson, *A note on hermitian forms,* Bulletin of the American Mathematical Society **46** (1940), 264–268.

[11] C. Jantzen, *Degenerate principal series for symplectic groups,* Memoirs of the American Mathematical Society **488** (1993), 1–111.

[12] C. Jantzen, *Degenerate principal series for orthogonal groups,* Journal für die reine und angewandte Mathematik **441** (1993), 61–98.

[13] M. Karel, *Values of certain Whittaker functions on p-adic groups,* Illinois Journal of Mathematics **26** (1982), 552–575.

[14] Stephen S. Kudla, *On the local theta-correspondence,* Inventiones mathematicae **83** (1986), 229–255.

[15] Stephen S. Kudla, *Splitting metaplectic covers of dual reductive pairs,* Israel Journal of Mathematics **87** (1994), 361–401.

[16] Stephen S. Kudla, *Some extensions of the Siegel-Weil formula,* preprint, RIMS, 1992.

[17] S. S. Kudla and S. Rallis, *On the Weil-Siegel formula*, Journal für die reine und angewandte Mathematik **387** (1988), 1–68.

[18] S. S. Kudla and S. Rallis, *Degenerate principal series and invariant distributions*, Israel Journal of Mathematics **69** (1990), 25–45.

[19] S. S. Kudla and S. Rallis, *Ramified degenerate principal series representations for* $Sp(n)$, Israel Journal of Mathematics **78** (1992), 209–256.

[20] S. S. Kudla and S. Rallis, *A regularized Siegel-Weil formula, the first term identity*, Annals of Mathematics **139** (1994), 1–80.

[21] Soo Teck Lee, *On some degenerate principal series representations of* $U(n,n)$, Thesis, Yale University, 1993.

[22] Soo Teck Lee and Chen-bo Zhu, *Degenerate principal series and the local theta correspondence*, preprint, 1994.

[23] C. Moeglin, M.-F. Vigneras and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adiques*, Lecture Notes in Mathematics **1291**, Springer-Verlag, Berlin–Heidelberg–New York, 1987, pp. 1–161.

[24] I. Piatetski-Shapiro and S. Rallis, *Rankin triple L functions*, Compositio Mathematica **64** (1987), 31–115.

[25] S. Rallis, *On the Howe duality conjecture*, Compositio Mathematica **51** (1984), 333–399.

[26] S. Rallis, *L-functions and the oscillator representation*, Lecture Notes in Mathematics **1245**, Springer-Verlag, Berlin–Heidelberg–New York, 1987, pp. 1–238.

[27] R. Ranga Rao, *On some explicit formulas in the theory of the Weil representation*, Pacific Journal of Mathematics **157** (1993), 335–371.

[28] F. Shahidi, *On certain L-functions*, American Journal of Mathematics **103** (1981), 297–355.

[29] F. Shahidi, *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Mathematical Journal **66** (1992), 1–41.

[30] W. J. Sweet, Jr., *The metaplectic case of the Weil-Siegel formula*, Ph.D. thesis, University of Maryland, College Park, 1990.

[31] W. J. Sweet, Jr., *A computation of the gamma matrix of a family of p-adic zeta integrals*, preprint

[32] M. Tadic, *On reducibility of parabolic induction*, preprint, 1993.

[33] J. T. Tate, *Fourier analysis in number fields and Hecke's zeta-functions*, in *Algebraic Number Theory* (J. W. S. Cassels and A. Frölich, eds.), Academic Press, New York, 1967.

[34] A. Weil, *Sur certains groupes d'operateurs unitaires,* Acta Mathematica **111** (1964), 143–211.

[35] A. Weil, *Sur la formule de Siegel dans la theorie des groupes classiques,* Acta Mathematica **113** (1965), 1–87.

[36] A. V. Zelevinsky, *Induced representations of reductive p-adic groups, II; on irreducible representations of* GL(n), Annales Scientifiques de l'École Normale Supérieure **4, t. 13** (1980), 165–210.

[37] Chen-bo Zhu, *Invariant distributions of classical groups,* Duke Mathematical Journal **65** (1992), 85–119.